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LECTURES ON

VECTOR ANALYSIS
&
VECTOR FIELDS

By
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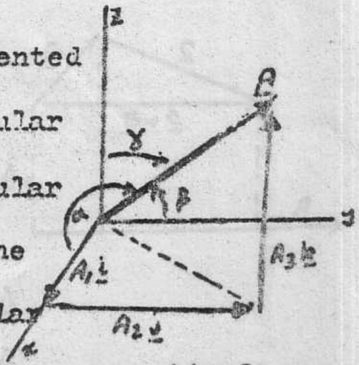
I-VECTOR ANALYSIS

## [1] 1.1 Components of a Vector:



Any vector $\underline{A}$ in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system. Let (A_1,A_2,A_3) be the rectangular coordinates of the terminal point of Vector A. The vectors $A_1\underline{i}$, $A_2\underline{j}$ and $A_3\underline{k}$ are called the rectangular component vectors of $\underline{A}$ in the x,y and z directions, respectively; $\underline{i},\underline{j},\underline{k}$ are the three unit vectors in these directions.

The sum or resultant of $A_1\underline{i}$, $A_2\underline{j}$ and $A_3\underline{k}$ is the vector $\underline{A}$, so that we can write

$$\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k} \qquad \ldots(1.1)$$

The magnitude of $\underline{A}$ is

$$A = |\underline{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \qquad \ldots(1.2)$$

In particular, the position vector, or radius vector, $\underline{r}$ from O to the point (x,y,z) is written

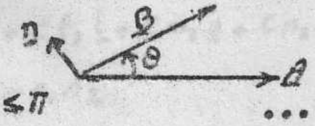
$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \qquad \ldots(1.3)$$

and has magnitude

$$r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \ldots(1.4)$$

1.2 Scalar Product:

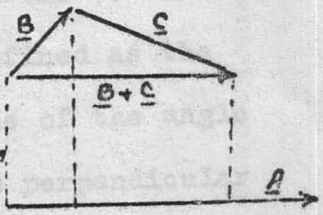
The dot (or scalar) product of two vectors $\underline{A}$ and $\underline{B}$, denoted by $\underline{A}\cdot\underline{B}$ (A dot B) is defined as the product of the magnitudes of $\underline{A}$ and $\underline{B}$ and the cosine of the angle between them. In symbols



$$\underline{A}\cdot\underline{B} = AB\cos\theta, \quad 0 \le \theta \le \pi \qquad \ldots(1.5)$$

is there a forbidden direction of motion in space (field) ?

Note that $\underline{A}.\underline{B}$ is a scalar and not a vector.



Ex.1.1: Prove $\underline{A}.(\underline{B}+\underline{C}) = \underline{A}.\underline{B} + \underline{A}.\underline{C}$

Let $\hat{\underline{a}}$ be a unit vector in the direction of $\underline{A}$, then

projection of $(\underline{B}+\underline{C})$ on $\underline{A}$=projection of $\underline{B}$ on $\underline{A}$ + projection of $\underline{C}$ on $\underline{A}$

or

$$(\underline{B}+\underline{C}).\hat{\underline{a}} = \underline{B}.\hat{\underline{a}} + \underline{C}.\hat{\underline{a}}$$

Multiplying by A, $(\underline{B}+\underline{C}).A\hat{\underline{a}} = \underline{B}.A\hat{\underline{a}} + \underline{C}.A\hat{\underline{a}}$

i.e.

$$(\underline{B}+\underline{C}).\underline{A} = \underline{B}.\underline{A} + \underline{C}.\underline{A}$$

Then by the commutative law the result follows.

Ex.1.2: Prove that for any vector $\underline{A}$,

$$\underline{A} = A(\cos\alpha\ \underline{i} + \cos\beta\ \underline{j} + \cos\gamma\ \underline{k})$$

where α, β, γ are the angles which $\underline{A}$ makes with x,y,z respectively.

Show that the angle between two vectors is given by

$$\theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are their direction cosines.

We have $\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$

Then $\underline{A}.\underline{i} = (A_1\underline{i} + A_2\underline{j} + A_3\underline{k}).\underline{i} = A_1$

Similarly $\underline{A}.\underline{j} = A_2$ & $\underline{A}.\underline{k} = A_3$

Thus $\underline{A} = (\underline{A}.\underline{i})\underline{i} + (\underline{A}.\underline{j})\underline{j} + (\underline{A}.\underline{k})\underline{k} = A\cos\alpha\,\underline{i} + A\cos\beta\,\underline{j} + A\cos\gamma\,\underline{k}$

Let $\underline{B} = Bl_1\underline{i} + Bm_1\underline{j} + Bn_1\underline{k}$ & $\underline{C} = Cl_2\underline{i} + Cm_2\underline{j} + Cn_2\underline{k}$

Then $\cos\theta = (\underline{B}.\underline{C})/BC = l_1l_2 + m_1m_2 + n_1n_2$

1.3 Vector Product:

The cross (or vector) product of $\underline{A}$ and $\underline{B}$ is the vector $\underline{C} = \underline{A} \wedge \underline{B}$ ($\underline{A}$ cross $\underline{B}$).The magnitude of $\underline{A} \wedge \underline{B}$ is defined as the product of the magnitudes of $\underline{A}$ and $\underline{B}$ and the sine of the angle between them.The direction of the vector $\underline{A} \wedge \underline{B}$ is perpendicular to the plane of $\underline{A}$ and $\underline{B}$,and such that $\underline{A},\underline{B},\underline{C}$ form a right-handed system.In symbols

$$\underline{A} \wedge \underline{B} = AB \sin\theta \, \underline{n} \,, \quad 0 \leq \theta \leq \pi \qquad \ldots(1.6)$$

where $\underline{n}$ is a unit vector indicating the direction of $\underline{A} \wedge \underline{B}$.

$\left| \underline{A} \wedge \underline{B} \right|$ = the area of a parallelogram with sides $\underline{A}$ and $\underline{B}$.

If $\underline{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{k}$ & $\underline{B} = B_1 \underline{i} + B_2 \underline{j} + B_3 \underline{k}$

then

$$\underline{A} \wedge \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \qquad \ldots(1.7)$$

If $\underline{A} \wedge \underline{B} = 0$,and $\underline{A}$ &$\underline{B}$ are not null vectors,then $\underline{A}$ &$\underline{B}$ are parallel.

Ex.1.3:Prove that

$$\underline{A} \wedge (\underline{B} + \underline{C}) = \underline{A} \wedge \underline{B} + \underline{A} \wedge \underline{C}$$

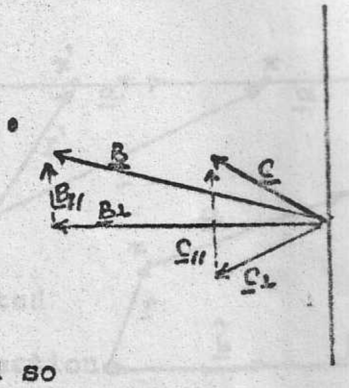
in the general case,where $\underline{A},\underline{B},\underline{C}$ are non-coplanar.

Resolve $\underline{B}$ into two component vectors,one perpendicular to $\underline{A}$ and the other parallel to $\underline{A}$,and denote them by $\underline{B}_\perp$ & $\underline{B}_\parallel$,respectively.

Then $$\underline{B} = \underline{B}_\perp + \underline{B}_\parallel$$

If θ is the angle between $\underline{A}$ & $\underline{B}$,then $|\underline{B}_\perp| = B \sin\theta$.Thus the magnitude of $\underline{A} \wedge \underline{B}_\perp$ is $AB \sin\theta$,the same as the magnitude of $\underline{A} \wedge \underline{B}$.

Hence $$\underline{A} \wedge \underline{B}_\perp = \underline{A} \wedge \underline{B}$$



Similarly, for C we get $\underline{A}\wedge\underline{C}_{\perp}=\underline{A}\wedge\underline{C}$

Also since $\underline{B}+\underline{C}=\underline{B}_{\perp}+\underline{B}_{\parallel}+\underline{C}_{\perp}+\underline{C}_{\parallel}$

it follows that $\underline{A}\wedge(\underline{B}_{\perp}+\underline{C}_{\perp})=\underline{A}\wedge(\underline{B}+\underline{C})$

Now $\underline{B}_{\perp}$ & $\underline{C}_{\perp}$ are vectors perpendicular to $\underline{A}$ and so

$$\underline{A}\wedge\underline{B}+\underline{A}\wedge\underline{C}=\underline{A}\wedge(\underline{B}+\underline{C})$$

Ex.1.4: Find the area of the triangle with vertices p(2,3,5), q(4,2,-1) and r(3,6,4).

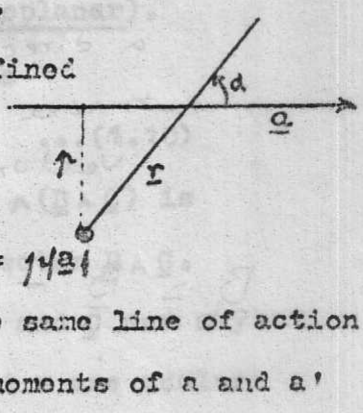
$\underline{pq} = 2\underline{i} - \underline{j} - 6\underline{k}$ and $\underline{pr} = \underline{i} + 3\underline{j} - \underline{k}$

$$\text{Area of triangle} = |\underline{pq}\wedge\underline{pr}|\ /2 = \left|\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & -6 \\ 1 & 3 & -1 \end{vmatrix}\right| /2 = \sqrt{426}/2$$

Ex.1.5: If $\underline{P}$ is the moment of $\underline{a}$ about a point O, prove that

$$|\underline{P}| = p|\underline{a}|$$

where p is the perpendicular distance from O to the line of action of $\underline{a}$. Hence show that the moments, about a point, of any two equal vectors, with the same line of action, are equal.

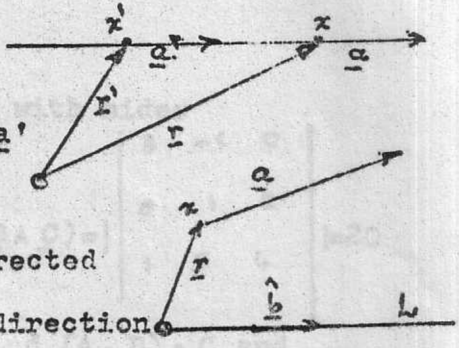


Now the moment of $\underline{a}$ about a point O is defined by the vector $\underline{P}$ given by $\underline{P} = \underline{r}\wedge\underline{a}$. Thus

$$|\underline{P}| = r\ a\ \sin\alpha,$$

where α is the angle between $\underline{r}$ and $\underline{a}$. Hence $|\underline{P}| = p|\underline{a}|$

Let $\underline{a}$ and $\underline{a}'$ be two equal vectors, with the same line of action and with origins x and x'. Let p and p' be the moments of a and a' about O. Then $\underline{P}=\underline{r}\wedge\underline{a}$, $\underline{P}'=\underline{r}'\wedge\underline{a}'$; $\underline{r}'=\underline{r}+\underline{xx}'$



Hence $\underline{P}' = \underline{r} \wedge \underline{a}' + \underline{xx}' \wedge \underline{a}'$

$= \underline{r} \wedge \underline{a}'$,since $\underline{xx}'$ is parallel to $\underline{a}'$

$= \underline{r} \wedge \underline{a}$,since $\underline{a}' = \underline{a}$

The moment Q of a vector $\underline{a}$ about a directed line L(with unit vector $\hat{\underline{b}}$ in the positive direction of the line) is

$Q = \hat{\underline{b}} \cdot (\underline{r} \wedge \underline{a})$,independent of the position of O on L.

1.4 Triple Products:

(i)The Scalar Triple Product(or box product) is defined as

$$\underline{A} \cdot (\underline{B} \wedge \underline{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \qquad \ldots(1.8)$$

and may be denoted by $[ABC]$.

It represents the volume of a parallelepiped,having $\underline{A}$,$\underline{B}$,$\underline{C}$ as edges.We have

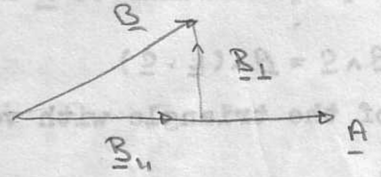
$$\underline{A} \cdot (\underline{B} \wedge \underline{C}) = \underline{B} \cdot (\underline{C} \wedge \underline{A}) = \underline{C} \cdot (\underline{A} \wedge \underline{B}) \qquad \ldots(1.9)$$

If $\underline{A} \cdot (\underline{B} \wedge \underline{C}) = 0$,the vectors lie in a plane(coplanar).

(ii)The Vector Triple Product is defined as

$$\underline{A} \wedge (\underline{B} \wedge \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C} \qquad \ldots(1.10)$$

By the definition of a cross product $\underline{q} = \underline{A} \wedge (\underline{B} \wedge \underline{C})$ is perpendicular to the vector $\underline{A}$ and also to the vector $\underline{B} \wedge \underline{C}$. Accordingly,the vector $\underline{q}$ lies in the plane of $\underline{B}$ and $\underline{C}$ and may be expressed in the form $\underline{q} = u\underline{B} + v\underline{C}$,where u and v are scalar multipliers.



$$\frac{\underline{A}\cdot\underline{B}}{\|\underline{A}\|}\ \frac{A}{|A|} = |B_\parallel| = \frac{\underline{A}\cdot\underline{B}}{|\underline{A}|^2} \quad \& \quad B_\parallel = \frac{\underline{A}\cdot\underline{B}}{\underline{A}\cdot\underline{A}}\underline{A} \qquad ①$$

$$\underline{B}_\perp = \frac{|\underline{A}\times\underline{B}|}{|A|}$$

$$\underline{B}_\perp = \underline{B} - \underline{B}_\parallel = \frac{(\underline{A}\times\underline{B})\times\underline{A}}{\underline{A}\cdot\underline{A}} \qquad ②$$

The proof is simple if we utilise the geometrical concept of a vector as a directed line segment i.e. not to use the component concept of a vector.

$$\underline{B}_\perp = \underline{B} - \frac{\underline{A}\cdot\underline{B}}{\underline{A}\cdot\underline{A}}\underline{A} = \frac{(\underline{A}\cdot\underline{A})\underline{B} - (\underline{A}\cdot\underline{B})\underline{A}}{\underline{A}\cdot\underline{A}}$$

Ex.1.6:Find the volume of a parallelepiped with sides $\underline{A} = 3\underline{i} - \underline{j}$, $\underline{B} = \underline{j} + 2\underline{k}$, $\underline{C} = \underline{i} + 5\underline{j} + 4\underline{k}$.

$$\text{Volume of parallelepiped} = [ABC] = \underline{A}\cdot(\underline{B}\wedge\underline{C}) = \begin{vmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 5 & 4 \end{vmatrix} = 20$$

Ex.1.7:If $\underline{A} = \underline{i} + \underline{j}$, $\underline{B} = 2\underline{i} - 3\underline{j} + \underline{k}$, $\underline{C} = 4\underline{j} - 3\underline{k}$, find $(\underline{A}\wedge\underline{B})\wedge\underline{C}$ and $\underline{A}\wedge(\underline{B}\wedge\underline{C})$ and prove that, in general, $(\underline{A}\wedge\underline{B})\wedge\underline{C} \neq \underline{A}\wedge(\underline{B}\wedge\underline{C})$.

$$\underline{A}\wedge\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{vmatrix}, \quad (\underline{A}\wedge\underline{B})\wedge\underline{C} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & -5 \\ 0 & 4 & -3 \end{vmatrix} = 23\underline{i} + 3\underline{j} + 4\underline{k}$$

& $\underline{A}\wedge(\underline{B}\wedge\underline{C}) = 8\underline{i} - 8\underline{j} + \underline{k}$

## [2] 1.5 Vectors in Analytic Geometry:

Consider a rectangular coordinate system in three dimensional space. Points are identified according to their locations, relative to the three mutually perpendicular planes. A given point p is then assigned the coordinates (x,y,z). If the position vector of p is $\underline{r}$, then $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

Consider two points $p_1(x_1,y_1,z_1)$ and $p_2(x_2,y_2,z_2)$, their position vectors are, respectively,

$$\underline{r}_1 = x_1\underline{i} + y_1\underline{j} + z_1\underline{k} \quad \& \quad \underline{r}_2 = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$$

The vector $\underline{P_1P_2} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$

The length of this vector is $\left|\underline{r}_2 - \underline{r}_1\right| = \left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\}^{1/2}$

The direction cosines of a directed line are the cosines of the angles between the positive direction of the line and the vectors $\underline{i},\underline{j},\underline{k}$.For the vector $\underline{p_1p_2}$,these are:

$$l=\frac{(x_2-x_1)}{|\underline{r}_2-\underline{r}_1|},\quad m=\frac{(y_2-y_1)}{|\underline{r}_2-\underline{r}_1|},\quad n=\frac{(z_2-z_1)}{|\underline{r}_2-\underline{r}_1|}$$

The quantities (x_2-x_1), (y_2-y_1) & (z_2-z_1) are the direction numbers.

(a) The Vector Equation of a Straight Line:

Let $\underline{r}_0$ denote the position vector of p_0 relative to some given origin and $\underline{r}$ denotes the position vector of an arbitrary point p.If $\underline{p_0p}$ is parallel to a given vector A,then

$$\underline{A}\wedge(\underline{r}-\underline{r}_0)=0 \qquad \ldots(1.11)$$

is the equation of the line through a specified point and having a specified direction.The vector $(\underline{r}-\underline{r}_0)$ is parallel to $\underline{A}$ if and only if it is a scalar multiple of $\underline{A}$,i.e. $(\underline{r}-\underline{r}_0)=t\underline{A}$ or $\underline{r}=\underline{r}_0+t\underline{A}$

As t varies from $-\infty$ to ∞,the point p traces out the entire line, each t corresponds to exactly one point on the line.

In terms of rectangular coordinates,the equation of the straight line is written in the form:

$$\frac{(x-x_0)}{l}=\frac{(y-y_0)}{m}=\frac{(z-z_0)}{n} \qquad \ldots(1.12)$$

Ex.1.3:Compute the distance between the two straight lines

$$\frac{x-1}{4}=\frac{y+2}{-3}=\frac{z-3}{1} \ \& \ \frac{x+1}{1}=\frac{y-2}{0}=\frac{z-1}{-3}$$

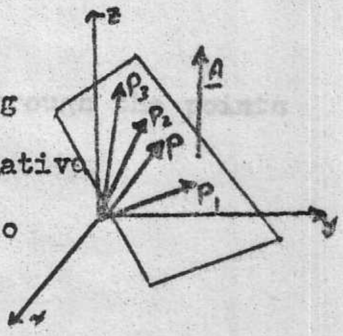
Choose two points $p_1(1,-2,3)$& $p_2(-1,2,1)$ on the first and second line,respectively. $\underline{p_2p_1}=2\underline{i}-4\underline{j}+2\underline{k}$.

Obtain the normal to the two lines

$$\underline{N} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & -3 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$

The distance between the two lines $= \underline{p_2p_1} \cdot \underline{N} = 28/\sqrt{259}$.

(b) The Vector Equation of a Plane:



Suppose that a plane is specified as passing through the point p_1, with position vector r_1 relative to a given origin O, and as being perpendicular to the direction of $\underline{A}$. Then a point p with position vector $\underline{r}$ lies on the plane if and only if the vector from p_1 to p, that is $\underline{r}-\underline{r}_1$, is perpendicular to $\underline{A}$, i.e.

$$\underline{A}\cdot(\underline{r}-\underline{r}_i)=0 \qquad \ldots(1.13)$$

This is the vector equation of the plane.

Let $\underline{r}_1 = x_1\underline{i}+y_1\underline{j}+z_1\underline{k}$, $\underline{r}_2 = x_2\underline{i}+y_2\underline{j}+z_2\underline{k}$ & $\underline{r}_3 = x_3\underline{i}+y_3\underline{j}+z_3\underline{k}$ be the position vectors of points

$$P_1(x_1,y_1,z_1) \quad , P_2(x_2,y_2,z_2) \quad \& \ P_3(x_3,y_3,z_3)$$

We assume that p_1, p_2, and p_3 do not lie on the same straight line, hence they determine a plane. Let $\underline{r} = x\underline{i}+y\underline{j}+z\underline{k}$ denote the position vector of any point p(x,y,z) on the plane. Consider vectors $\underline{P_1P_2} = \underline{r}_2-\underline{r}_1$, $\underline{P_1P_3} = \underline{r}_3-\underline{r}_1$ & $\underline{P_1P} = \underline{r}-\underline{r}_1$ which all lie on the plane. Then $\underline{P_1P}\cdot(\underline{P_1P_2}\wedge\underline{P_1P_2}) = 0$

$$\text{or} \quad (\underline{r}-\underline{r}_1)\cdot[(\underline{r}_2-\underline{r}_1)\wedge(\underline{r}_3-\underline{r}_1)] = 0 \qquad \ldots(1.14)$$

which is the equation of the plane passing through p_1, p_2, p_3.

In terms of rectangular coordinates, Eq.(1.14) becomes:

$$\begin{vmatrix} (x-x_1) & (y-y_1) & (z-z_1) \\ (x_2-x_1) & (y_2-y_1) & (z_2-z_1) \\ (x_3-x_1) & (y_3-y_1) & (z_3-z_1) \end{vmatrix} = 0 \qquad \ldots(1.15)$$

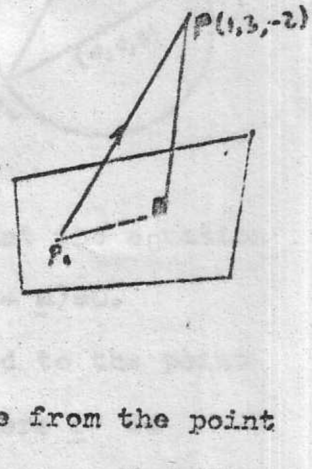
Ex.1.9: Find an equation of the plane passing through the points

$P_1(3,1,-2)$, $P_2(-1,2,4)$, $P_3(2,-1,1)$

The equation of the plane is

$$\begin{vmatrix} (x-3) & (y-1) & (z+2) \\ (-1-3) & (2-1) & (4+2) \\ (2-3) & (-1-1) & (1+2) \end{vmatrix} = 0 \qquad \text{or } 5x+2y+3z=0$$

Ex.1.10: Compute the distance from the point $(1,3,-2)$ to the plane $4x-3y+12z=10$

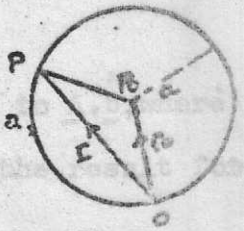


Choose any point p_o on the plane. The desired distance is the length of projection of the vector $\underline{p_o p}$ along the normal to the plane. Let $p_o(1,2,1)$, then $\underline{p_o p} = \underline{j} - 3\underline{k}$,

$$\underline{n} = (4\underline{i} - 3\underline{j} + 12\underline{k})/13$$

is the unit vector normal to the plane. The distance from the point $(1,3,-2)$ to the plane is $\underline{p_o p} \cdot \underline{n} = -3$. Negative sign means that the point is below the plane.

(C) The Vector Equation of a Sphere:



Consider a sphere with center at p_0 and radius a, then

$$|\underline{r}-\underline{r}_0| = a \qquad \ldots(1.16)$$

is the condition satisfied by those points on the sphere. Squaring both sides we obtain,

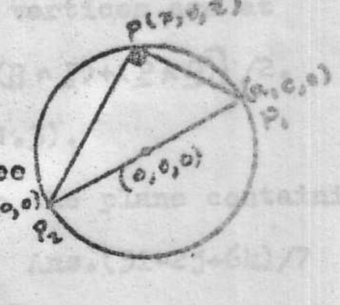
$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = a^2 \qquad \ldots(1.17)$$

which is the normal equation of a sphere in rectangular coordinates.

Ex.1.11:Show that vectors drawn from the points (a,0,0) and (-a,0,0) to any point on the sphere r=a are perpendicular. Observe then that the equation of the sphere can be written:

$$(\underline{r} - \underline{a}).(\underline{r} + \underline{a}) =0,$$

where $\underline{a}$ is the position vector of (a,0,0).



Let p_1(a,0,0),p_2(-a,0,0) and p(x,y,z) be three points on the sphere,then $\underline{P_1P} = (x-a)\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ & $\underline{P_2P} = (x+a)\underline{i} + y\,\underline{j} + z\,\underline{k}$. or $\underline{P_1P} \cdot \underline{P_2P} = x^2+y^2+z^2-a^2 = 0$

Thus the two vectors are perpendicular.This shows that the equation of the sphere can be written in the form $(\underline{r} - \underline{a}).(\underline{r} + \underline{a})=0$.

In the above example if the origin is translated to the point (a,0,0).the position vector $\underline{r}$ is replaced by $\underline{r}'+\underline{a}$,where $\underline{r}'$ is the new position vector.The corresponding equation in rectangular coordinates,centered at this new origin is:

$\underline{r}'.(\underline{r}'+2\underline{a})=0$ or $x^2+y^2+z^2+2ax =0$.

Problems-Vector Analysis

1.1 prove that the projection of $\underline{A}$ on $\underline{B}$ is equal to $\underline{A}.\hat{\underline{b}}$, where $\hat{\underline{b}}$ is a unit vector in the direction of $\underline{B}$. Verify the result fot

$\underline{A} = 4\underline{i} - 4\underline{j} + 2\underline{k}$ and $\underline{B} = 6\underline{i} + 3\underline{j} - 6\underline{k}$

1.2 Find the acute angle between the diagonals of a quadrilateral having vertices at (0,0,0),(3,2,0),(4,6,0) and (1,3,0).

Ans. $\theta = \cos^{-1} 1/\sqrt{65}$

1.3 Prove that the area of a parallelogram with sides $\underline{A}$ and $\underline{B}$ is $\underline{A} \wedge \underline{B}$ .Hence find the area of the triangle with vertices at p(2,-3,1),Q(1,-1,2) and R(-1,2,3). Ans. $\sqrt{3}/2$

1.4 Prove that the area of the triangle whose vertices are at P,Q,R is the magnitude of the vector $[(\underline{Q} \wedge \underline{R})+(\underline{R} \wedge \underline{P})+(\underline{P} \wedge \underline{Q})]/2$. Check your result using the points given in (1.3).

1.5 Determine the unit vector perpendicular to the plane containing

$\underline{A} = 2\underline{i} - 6\underline{j} - 3\underline{k}$ ahd $\underline{B} = 4\underline{i} + 3\underline{j} + \underline{k}$. Ans. $(3i-2j+6k)/7$

1.6 Find the equation of the plane passing through (2,-1,-2), (-1,2,-3) and (4,1,0). Determine the unit vector perpendicular to that plane.

Ans. $(2x+2y-3z=9)$

$(2i+2j-3k)/\sqrt{17}$

1.7 Obtain the equation of a plane containing the point (1,-2,0) and the two vectors $\underline{A}=\underline{i} - \underline{j} - 2\underline{k}$ and $\underline{B} = 2\underline{i} + \underline{k}$.

Ans. $(x+5y-2z=-9)$

1.8 Determine a unit vector along the line of intersection of the planes $x - y + 2z = 3$ and $2x + y - 4z = 1$

Ans. $(2\underline{i}+8\underline{j}+3\underline{k})/\sqrt{77}$

1.9 Write the equation of the line perpendicular to $2\underline{i} - \underline{j} + \underline{k}$ and parallel to the plane $2x + y - 4z = 1$ and passing through the point $(1,0,-3)$.

Ans. $(x-13/4)/3=(y-15/2)/10=z/4$

1.10 Determine the equation of the plane tangent to the sphere $r=13$ at the point $(3,-4,12)$. Ans. $(3x-4y+12z=169)$.

II-DERIVATIVES & INTEGRALS ofVECTORS

2.1 Differentiation & Integration of Vector Function:

If to each value, assumed by a scalar variable u, there cooresponds a vector $\underline{A}(u)$, or briefly $\underline{A}$, then $\overset{\underline{A}(u)}{\underline{A}(u)}$ is called a (vector) function of u. The derivative of $\underline{A}(u)$ is defined as:

$$\frac{d\underline{A}}{du} = \lim_{\Delta u \to 0} \frac{\underline{A}(u+\Delta u) - \underline{A}(u)}{\Delta u} \qquad \ldots(2.1)$$

provided this limit exists.

$\underline{A}(u)$ is a vector valued function of u.

If $\underline{A}(u) = A_1(u)\underline{i} + A_2(u)\underline{j} + A_3(u)\underline{k}$, then

$$\frac{d\underline{A}}{du} = \frac{dA_1}{du}\underline{i} + \frac{dA_2}{du}\underline{j} + \frac{dA_3}{du}\underline{k} \qquad \ldots(2.2)$$

Similarly we can define higher derivatives.

If $\overset{(i)}{\phi}(u)$ is a scalar function, while $\overset{(ii)}{\underline{A}}(u)$ and $\overset{(ii)}{\underline{B}}(u)$ are vector functions, then

$$\frac{d}{du}(\phi\underline{A}) = \phi\frac{d\underline{A}}{du} + \frac{d\phi}{du}\underline{A} \qquad \ldots(2.3)$$

$$\frac{d}{du}(\underline{A}\cdot\underline{B}) = \underline{A}\cdot\frac{d\underline{B}}{du} + \underline{B}\cdot\frac{d\underline{A}}{du} \qquad \ldots(2.4)$$

$$\frac{d}{du}(\underline{A}\wedge\underline{B}) = \underline{A}\wedge\frac{d\underline{B}}{du} + \frac{d\underline{A}}{du}\wedge\underline{B} \qquad \ldots(2.5)$$

The indefinite integral of $\underline{A}(u)$ is defined as

$$\int \underline{A}(u)\,du = \underline{i}\int A_1(u)\,du + \underline{j}\int A_2(u)\,du + \underline{k}\int A_3(u)\,du \ldots(2.6)$$

If there exists a vector function $\underline{B}(u)$ such that

$$\underline{A}(u) = \frac{d}{du}\left\{\underline{B}(u)\right\}$$

then $\left\{ \int \underline{A}(u)\,du = \int \frac{d}{du}\left[\underline{B}(u)\right]du = \underline{B}(u) + \underline{C} \right\}$ ...(2.7)

(where $\underline{C}$ is an arbitrary constant vector independent of u.)

The definite integral between limits $u=\alpha$ and $u=\beta$ is given by

$$\int_\alpha^\beta \underline{A}(u)\,du = \int_\alpha^\beta \frac{d}{du}\left[\underline{B}(u)\right]du = \underline{B}(\beta) - \underline{B}(\alpha) \qquad \ldots(2.8)$$

Ex.2.1: If $\phi(x,y,z)=x^2yz$ and $\underline{A}=3x^2y\underline{i}+yz^2\underline{j}-xz\underline{k}$, find

$\frac{\partial^2}{\partial y\partial z}(\phi A)$ at the point $(1,-2,-1)$.

$$\phi A=3x^4y^2z\,\underline{i}+x^2y^2z^3\,\underline{j}-x^3yz^2\,\underline{k}$$

$$\frac{\partial^2}{\partial y\partial z}(\phi\underline{A})=6x^4y\,\underline{i}+6x^2yz^2\,\underline{j}-2x^3z\,\underline{k}$$

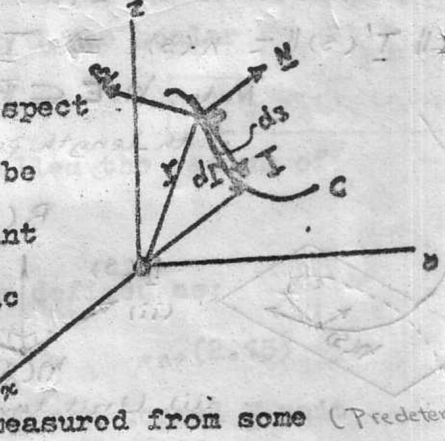
If $x=1, y=-2, z=-1$ this becomes $-12\underline{i}-12\underline{j}+2\underline{k}$

Ex.2.2: Evaluate $\int_{u=1}^{2}\underline{A}(u)\,du$ if $\underline{A}(u)=(3u^2-1)\underline{i}+(2u-3)\underline{j}+(6u^2-4u)\underline{k}$

The given integral equals $\int_{u=1}^{2}\{(3u^2-1)\underline{i}+(2u-3)\underline{j}+(6u^2-4u)\underline{k}\}\,du$

$$=\{(u^3-u)\underline{i}+(u^2-3u)\underline{j}+(2u^3-2u^2)\underline{k}\}\Big|_{u=1}^{2}=6\underline{i}+8\underline{k}$$

(a) 2.2 Differentiation with respect to time:



Let the position of the particle with respect to the origin of a cartesian reference frame be denoted by $\underline{r}(u)$. As u changes, (the terminal point of $\underline{r}$ describes a space curve having parametric equations: $x=x(u)$, $y=y(u)$, $z=z(u)$.)

(i) If the parameter u is the arc length s, measured from some (Predetermined) fixed point on the curve, then

$$d\underline{r}/ds=\underline{T} \qquad \ldots(2.9)$$

is the unit tangent vector.

(ii) If u is the time t, then

$$d\underline{r}/dt=\underline{V} \qquad \ldots(2.10)$$

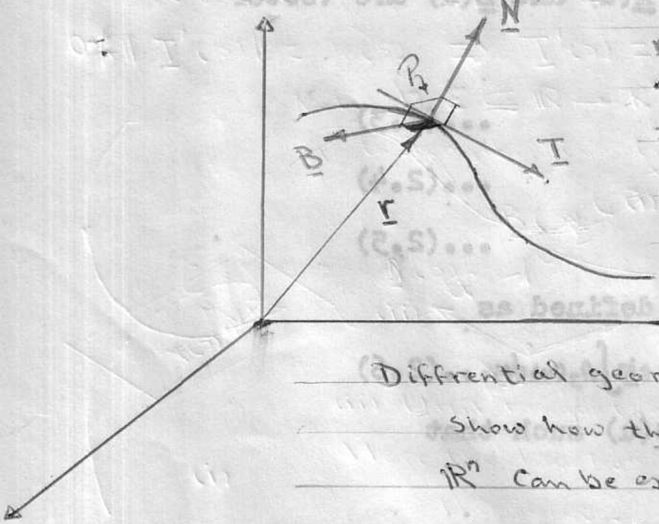
$(\partial(\phi A)/\partial y) = (x^2yz)(3x^2\underline{i} + z^2\underline{j}) +$
$(3x^2y\underline{i} + yz^2\underline{j} - xz\underline{k})(zx^2) \Rightarrow$

$(\partial(\partial(\phi\underline{A})/\partial y)/\partial z) = (x^2yz)(2z\underline{j}) + (3x^2\underline{i} + z^2\underline{j})(x^2y)$
$+ (3x^2y\underline{i} + yz^2\underline{j} - xz\underline{k})(x^2) + (zx^2)(2yz\underline{j} - x\underline{k}) =$

* * *

$\int_{u=1}^{2} \underline{A}(u)\,du = (\frac{3}{3}u^3 - u)\underline{i} + (\frac{2}{2}u^2 - 3u)\underline{j} + (\frac{6}{3}u^3 - \frac{4}{2}u^2)\underline{k}\Big|_1^2$
$[(8-2)-0]\underline{i} + [(4-6)-(1-3\times1)]\underline{j} + [(16-8)-(2-2)]\underline{k} =$
$6\underline{i} + 0\underline{j} + 8\underline{k} = 6\underline{i} + 8\underline{k}$

* * *



$\underline{r}(s+\Delta s) - \underline{r}(s) = d\underline{r}(s)$

$\underline{r} + \delta\underline{r} = \underline{r}' \Rightarrow \delta\underline{r} \neq \underline{r} - \underline{r}' \Rightarrow |\delta\underline{r}| = \delta s$

$\Rightarrow$ taking the diffrential form $\Rightarrow |d\underline{r}| = ds$
hence $d\underline{r}/ds$ is a unit vector
tangent to the Curve $= \underline{T}$.

Diffrential geometry :- In this section we shall show how the twisting and turning of a Curve in $\mathbb{R}^n$ Can be expressed in terms of the derivatives of a path-length parametrization.

is the velocity of the particle. We have

$$\underline{V} = d\underline{r}/dt = (d\underline{r}/ds)(ds/dt) = \underline{T}\,(ds/dt) = \underline{T}\,v \qquad \ldots(2.11)$$

where

v= ds/dt is the speed of the particle.

(i) Since T is a unit tangent vector, we have

$$\underline{T}\cdot\underline{T} = 1 \Rightarrow \frac{d}{ds}(\underline{T}\cdot\underline{T}) = 0 \qquad \ldots(2.12)$$

Then, differentiating with respect to s, we obtain

$$\underline{T}\cdot(d\underline{T}/ds) + (d\underline{T}/ds)\cdot\underline{T} = 2\underline{T}\cdot(d\underline{T}/ds) = 0 \Rightarrow \boxed{(d\underline{T}/ds) = d(d\underline{r}/ds)/ds = \left(\frac{d^2\underline{r}}{ds^2}\right)}$$

or

$$\underline{T}\cdot(d\underline{T}/ds) = 0 \qquad \ldots(2.13)$$

which states that $d\underline{T}/ds$ is normal, i.e. perpendicular to $\underline{T}$.

If $\underline{N}$ is a unit vector in the direction of $d\underline{T}/ds$, we have

why (dT/ds) dose lie in the plane of T and ⊥ T!

$$d\underline{T}/ds = K\,\underline{N} \qquad \text{Go To } ⊛ \qquad \ldots(2.14)$$

Notice (dT) represents change of direction and (ds) change in length ∴ $\left\|\frac{d\underline{T}}{ds}\right\| \neq 1$ in general; $\left\|\frac{d\underline{T}}{dT}\right\| = 1$ where $\|d\underline{T}\| = dT$

and we call N the unit principal normal to C. The scalar

$$(\text{kappa}) \quad K = |d\underline{T}/ds| = \left(\frac{\text{length}/\text{length}}{\text{length}}\right)$$

is called the curvature at (S), while R =1/K is called the radius of curvature. (ρ). ρ = 1/K

The unit Binormal B to the curve C is defined as:

$$\underline{B} = \underline{T}\wedge\underline{N} \quad \left(\begin{matrix}\text{a R.H system}\\ \text{of coordinates}\end{matrix}\right) \qquad \ldots(2.15)$$

(ii) Using the definitions of the tangent $\underline{T}$ and normal $\underline{N}$ unit vectors (Eqs. (2.11) & (2.14)), the acceleration of the particle is

$$\underline{a} = (d\underline{V}/dt) = d(\underline{T}\,v)/dt = d(\underline{T}\,ds/dt)/dt$$

⊙ $d\underline{T}/dt = (d\underline{T}/ds)\cdot(ds/dt) = (\underline{v}/\rho)\,\underline{N}$

Then

$$\underline{a} = \overset{\odot}{(d\underline{T}/dt)}(ds/dt) + \underline{T}\,d^2s/dt^2$$

or

$$\underline{a} = (\underline{N}/R)(ds/dt)^2 + \underline{T}\,d^2s/dt^2 \qquad \ldots(2.16)$$

$$\underline{a} = v\left(\frac{dv}{ds}\right)\underline{T} + \frac{v^2}{\rho}\underline{N} \quad ; \ \underline{V} = v\,\underline{T} \qquad [1]$$

Now, what is the direction of $(d\underline{T})$, hence from ⊛ $d\underline{T}$ tends to be orthogonal to $\underline{T}(S)$ which means that it tends to $\underline{N}$ at (S). Therefore $(d\underline{T}/ds)$ is parallel to $\underline{N}$. Notice $dS = \rho\,d\theta$ and as $\Delta S \to 0 \Rightarrow \Delta\theta \to 0$. $\Rightarrow \|d\underline{T}/ds\| = \kappa\|\underline{N}\| = \kappa$ (i.e. from)

(1) For a helix $T(s) = h'(s) = \frac{1}{c}\left(a, -b\omega \sin\frac{\omega s}{c}, b\omega \cos\frac{\omega s}{c}\right)$.

The subspace spanned by the orthogonal unit vector $T(s)$ and $N(s)$ in $\mathbb{R}^3$ is the plane

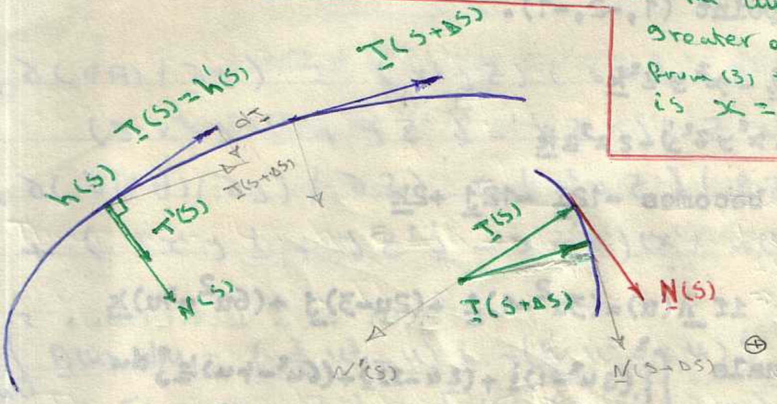
(2) $b\omega x + a\left(\sin\frac{\omega s}{c}\right)y - a\left(\cos\frac{\omega s}{c}\right)z = 0$.

The osculating plane to h at s is the plane parallel to this through $h(s)$,

(3) given by $b\omega x + a\left(\sin\frac{\omega s}{c}\right) - a\left(\cos\frac{\omega s}{c}\right)z = \frac{ab\omega s}{c}$

suppose now that $a = 0$. The image of h is then a circle of radius b in the x, z plane.

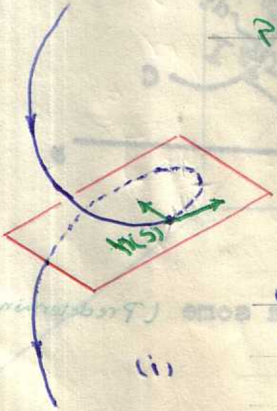
$\therefore \ \kappa = \frac{b\omega^2}{c} = \frac{1}{b}$, reflecting the fact that the curvature of a circle becomes greater as the radius becomes smaller. from (3) the osculating plane here when $a = 0$, is $x = 0$, the y, z plane itself.


$\underline{T}(S+\Delta S)$
$\underline{T}(S) = \underline{h}'(S)$
$d\underline{T}$
$\underline{T}(S+\Delta S)$
$h(S)$
$\underline{T}'(S)$
$\underline{N}(S)$
$\underline{T}(S)$
$\underline{T}(S+\Delta S)$
$\underline{N}(S)$
$\underline{N}'(S)$
$\underline{N}(S+\Delta S)$
⊕

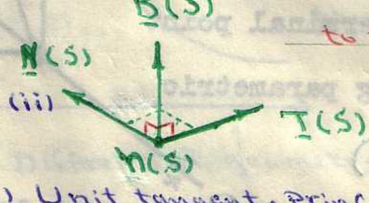
⊕ The vector $(d\underline{T}/dS)$ indicates the way in which the unit tangent vector $\underline{T}$ is changing direction near S. The larger the value of $\|d\underline{T}/dS\| = \kappa$, the greater is the 'curving' of C at S.

$0 \neq \|\underline{T}'(S)\| = \kappa(S) \Rightarrow \underline{T}'(S) = \kappa(S)\,\underline{N}(S)$.

Note: $h: E \subseteq \mathbb{R} \to \mathbb{R}^n$ be a twice-differentiable (in this case) path-length parametrization of the curve $C = h(E)$.



(i)



(ii)

(i) The plane in $\mathbb{R}^n$ through $h(S)$ parallel to the subspace spanned by $\underline{T}(S)$ and $\underline{N}(S)$ is called the osculating plane of h at S.

(ii) Unit tangent, principle normal and binormal at $h(S)$. Here the direction of $\underline{N}(S)$ is towards the concave side of the curve.

(I) To get the subspace spanned by $\underline{T}$ and $\underline{N}$ in $\mathbb{R}^3$ is which is the plane passing $\underline{r}_1 = 0 \Rightarrow (\underline{r} - \underline{r}_1)\cdot\underline{B} = 0$, $\underline{B} = \underline{T} \times \underline{N}$

(II) The osculating plane to h at S is the plane parallel to this through $h(S) \Rightarrow (\underline{r} - \underline{h}(S))\cdot\underline{B} = 0$

Since $\boxed{d\underline{T} = \underline{T}(S+\Delta S) - \underline{T}(S)}$; in general C in $\mathbb{R}^3 \Rightarrow$ we have two planes as in (ii) above 1st of $\underline{N}$ and $\underline{T}$, 2nd of $\underline{B}$ and $\underline{T}$ $\Rightarrow$ as $\Delta S \to 0 \Rightarrow$ the $\underline{B}$ at $(S+\Delta S) \to$ the $\underline{B}$ at (S).

Now, even though, suppose that $(d\underline{T})$ still exists, which means that we want to describe something which is a characteristic of a ΔS of a curve in a plane, i.e. " " " "
" " " " " " a point.

We thus see that the acceleration of the particle consists of two terms. The first term is the centripetal acceleration (v^2/ρ) of the particle and depends on the radius of curvature R of the particle and the square of the speed; the second term $(dv/dt) = v\,(dv/ds)$ depends upon the rate of change of speed of the particle and is directed along the tangent to the path of the particle.

Ex.2.3: Given a space curve C with position vector

$$\underline{r} = 3\cos 2t\,\underline{i} + 3\sin 2t\,\underline{j} + (8t-4)\underline{k}$$

i) Find a unit tangent vector $\underline{T}$ to the curve. Show that $\underline{V} = v\,\underline{T}$, where $\underline{V}$ and v are the particle's velocity and speed respectively.

ii) Find the curvature, unit principal $\underline{N}$ and the binormal $\underline{B}$ at a point of the space curve C.

A tangent vector to C is:

$$d\underline{r}/dt = -6\sin 2t\,\underline{i} + 6\cos 2t\,\underline{j} + 8\underline{k}$$

The magnitude of this vector is

$$|d\underline{r}/dt| = ds/dt = \sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2 + (8)^2} = 10$$

Then a unit tangent vector to C is

$$\underline{T} = (d\underline{r}/dt)/|d\underline{r}/dt| = (d\underline{r}/dt)/(ds/dt) = d\underline{r}/ds$$

$$= -3/5\sin 2t\,\underline{i} + 3/5\cos 2t\,\underline{j} + 4/5\,\underline{k}$$

Hence

$$\underline{V} = d\underline{r}/dt = -6\sin 2t\,\underline{i} + 6\cos 2t\,\underline{j} + 8\underline{k} = 10\underline{T}$$

Note that in this case the speed of the particle, along the curve, is constant.

Oh Fortuna

By Karmiena Bourana

$$d\underline{T}/ds = (d\underline{T}/dt)/(ds/dt) = -3(\cos 2t\ \underline{i} + \sin 2t\ \underline{j})/25$$

The curvature $K = |d\underline{T}/ds| = 3/25$, radius of curvature $R = 1/K = 25/3$

The Normal $\underline{N} = (d\underline{T}/ds)/K = R(d\underline{T}/ds) = -(\cos 2t\ \underline{i} + \sin 2t\ \underline{j})$

The Binormal $\underline{B} = \underline{T} \wedge \underline{N} = (4 \sin 2t\ \underline{i} - 4 \cos 2t\ \underline{j} + 3\underline{k})/5$

(i) Ex.2.4: If $\underline{T}$, $\underline{N}$ and $\underline{B}$ are the unit tangent vector, unit principal normal and unit binormal to a space curve $\underline{r} = \underline{r}(u)$, assumed differentiable, prove that

$$d\underline{B}/ds = -\tau \underline{N} \quad \text{and} \quad d\underline{N}/ds = \tau \underline{B} - K \underline{T}$$

where K is the curvature and τ is the torsion. ("torsion" twisting or turning)

Since ⓪ $\underline{B} = \underline{T} \wedge \underline{N}$; $d\underline{B}/ds = \underline{T} \wedge (d\underline{N}/ds) + \overset{= K\underline{N}}{(d\underline{T}/ds)} \wedge \underline{N} = \underline{T} \wedge (d\underline{N}/ds)$

this identity contains two differentials

⇒ we want to know the orientation of them w.r. to $\underline{T}$ & $\underline{B}$ & $\underline{N}$ ⇒ ① ⇒ ②

Then $\underline{T} \cdot (d\underline{B}/ds) = \underline{T} \cdot (\underline{T} \wedge d\underline{N}/ds) = 0$

Thus $\underline{T}$ is perpendicular to $d\underline{B}/ds$. ① $\underline{T}$ & $\underline{B}$ $\perp (d\underline{B}/ds)$

But $\underline{B} \cdot \underline{B} = 1$, thus $\underline{B} \cdot (d\underline{B}/ds) = 0$, i.e. $\underline{B}$ is perpendicular to $d\underline{B}/ds$. ②

can (τ) be (+ve) or (−ve)

⇒ why the (−ve) sign

Thus $$d\underline{B}/ds = -\tau \underline{N}$$

R-H system

Let $\underline{N} = \underline{B} \wedge \underline{T}$; $d\underline{N}/ds = \underline{B} \wedge d\underline{T}/ds + d\underline{B}/ds \wedge \underline{T}$

$$= \underline{B} \wedge K\underline{N} \quad - \tau \underline{N} \wedge \underline{T}$$

or $$d\underline{N}/ds = -K\underline{T} + \tau \underline{B}$$

The relations for $d\underline{T}/ds$, $d\underline{B}/ds$ and $d\underline{N}/ds$ are called the Frenet-Sérret formulae.

$d\underline{B}/ds = -\tau \underline{N}$ from which we can infer that we want to know what happens when (s) vary and also $\underline{B}$ varies its direction i.e. C doesn't lie in a plane ⇒ we ~~what~~ want to know the behavior of $(d\underline{B}/ds)$ from ⇒ ⓪ ⇒ ① ⇒ ② above

i.e. we ~~[illegible]~~ want to describe something which is a characteristic of C at Δs as $\Delta s \to 0$ i.e. of a point; as $\underline{B}$ varies its direction ~~[illegible]~~; in a plane ⊥ to the plane of $\underline{T}$ and $\underline{N}$.

a- **Theorem:** Let $h : D \subseteq \mathbb{R} \to \mathbb{R}^n$ be a diffrentiable function s.t $\|h(t)\| = 1$ $\forall t \in D$, Then $h(t)$ and $h'(t)$ are orthogonal $\forall t \in D$.

Proof: The image of h lies on the unit sphere center 0 in $\mathbb{R}^n$. $\forall t \in D$ $\quad h(t) \cdot h(t) = \|h(t)\|^2 = 1$ On diffrentiating the two sides, then $2h(t) \cdot h'(t) = 0$ $\forall t \in D$. Therefore the tangent vector $h'(t)$ is orthogonal to the radius vector $h(t)$.

Theorem-2- is an example of how the functions κ, T and N can be used to study the geometry of curves in $\mathbb{R}^3$. Theorem (a) implies that the The vector $T'(s) = \kappa(s) N(s)$ is orthogonal to $T(s)$ and similary that $N(s)$ is orthogonal to $N'(s)$. It is therefore possible that $N'(s)$ and $T(s)$ are in the same direction.

1- **Theorem:-** Let $h : E \subseteq \mathbb{R} \to \mathbb{R}^n$ be a twice diffrentiable path-length parametrization. Then $\kappa(s) = 0$ for all $s \in E$ if and only if $h(E)$ is a subset of a straight line in $\mathbb{R}^n$

2- **Theorem:-** Let $h : E \subseteq \mathbb{R} \to \mathbb{R}^3$ be a three times diffrentiable path length parametrization of a curve C in $\mathbb{R}^3$ with $\kappa(s) \neq 0$ for all $s \in E$. The curve C lie in a plane in $\mathbb{R}^3$ if and only if $N'(s)$ is a multiple of $T(s)$ for all $s \in E$.

Proof:- Diffrentiating the identity $N(s) \cdot T(s) = 0$ gives

$$(N'(s) \cdot T(s)) + (N(s) \cdot T'(s)) = 0.$$

Hence $\left[N'(s) \cdot T(s) = -\kappa(s),\ s \in E \quad (1) \right]$

Suppose that $N'(s)$ is a multiple of $T(s)$. It follows from (1) that See ⊕ Page(15)

$\left[N'(s) = -\kappa(s) T(s),\ s \in E. \quad (2) \right]$. Choose any $q \in E$. We shall prove that C lies in the osculating plane of h at q. It is enough to show that for all $s \in E$ the vector $h(s) - h(q)$ lies in that plane. This will follow if we can show that, $\forall s \in E$ $\left[(h(s) - h(q)) \cdot (T(q) \times N(q)) = 0 \quad (3) \right]$

Put $\underline{B}(s) = \underline{T}(s) \times \underline{N}(s)$. Then it follows from $\underline{T}' = \kappa \underline{N}$ and (2) that $\underline{B}'(s) = 0$.

So $\underline{B}(s) = \text{Const.} = \underline{B}(q)$. But $\underline{T}(s) \cdot \underline{B}(s) = 0$, and therefore

$\left[\underline{T}(s) \cdot (\underline{T}(q) \times \underline{N}(q)) = 0 \quad \forall s \in E \quad (4) \right]$ Integrating (4), we obtain

$\underline{h}(s) \cdot (\underline{T}(q) \times \underline{N}(q)) = \text{Const} = h(q) \cdot (T(q) \times N(q))$, and (3) is proved

The proof of the converse proposition, that if C lies in a plane then $N'(s)$ is a multiple of $T(s)$ for all $s \in D$, is left to the reader,

10 It follows from theorem-2- and in particular from (2) that the norm of the vector $[N'(s) + \kappa(s) T(s)]$ indicates whether or not the curve has any torsion - that is to say whether or not the curve twists out of its osculating planes.

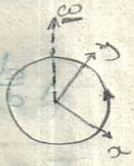
2.3 Application to Mechanics:

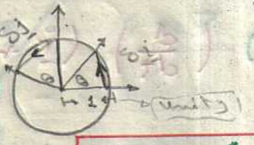
(i) Rotating Axes: Consider a small particle on a rotating plane sheet of cardboard, rotating about a fixed point O with a uniform angular velocity w. Let the position vector of the particle be

($\underline{\omega}$) rotating angular velocity

$$\underline{r} = x\underline{i} + y\underline{j}$$

where $\underline{i}$ and $\underline{j}$ are unit vectors in the x and y directions that (i.e. $\underline{i}$ & $\underline{j}$) rotate with the cardboard. The velocity V relative to fixed space is:





$$\underline{V} = d\underline{r}/dt = d(x\underline{i} + y\underline{j})/dt \quad = (\dot{x} - \omega y)\underline{i} + (\dot{y} + \omega x)\underline{j} \; (1)$$

$$= (dx/dt)\underline{i} + x(d\underline{i}/dt) + (dy/dt)\underline{j} + y(d\underline{j}/dt)$$

$\|\delta\underline{i}\| = \delta s$

$\delta s = r \times \delta\theta$

when $r = 1$

$\Rightarrow \delta s = \delta\theta$

$\Rightarrow \frac{d\underline{i}}{ds} =$

$\lim_{\delta s \to 0} \frac{\delta\underline{i}}{\delta s} = \underline{j}$

when $\frac{ds}{dt} = \frac{d\theta}{dt} = \dot{\theta}$ i.e. not const.

Now, since $\underline{i}$ and $\underline{j}$ are unit vectors rotating with angular velocity (ω) w, we get

$d\underline{i}/dt = (d\underline{i}/ds)\left(\frac{ds}{dt}\right) \Rightarrow$ when $\boxed{\frac{ds}{dt} = \omega}$

$$d\underline{i}/dt = w\,\underline{j} \quad \text{and} \quad d\underline{j}/dt = -w\,\underline{i} \qquad \ldots(2.17)$$

$\|\delta\underline{j}\| = \delta s \Rightarrow \lim_{\delta s \to 0} \frac{\delta\underline{j}}{\delta s}\left(\frac{ds}{dt}\right)$ $\underline{j}' - \underline{j} = \delta\underline{j}$

as $\delta s \to 0 \Rightarrow \delta\underline{j} \to -\underline{i} \Rightarrow \frac{d\underline{j}}{ds} = -\omega\underline{i}$

Hence

$$\underline{V} = (\dot{x} - w\,y)\,\underline{i} + (\dot{y} + w\,x)\,\underline{j} \qquad \ldots(2.18)$$

$$\underline{V} = (\dot{x} - \omega y)\underline{i} + (\dot{y} + \omega x)\underline{j} \quad (1)$$

and

$$\underline{a} = d\underline{V}/dt = (\ddot{x} - 2w\dot{y} - w^2x)\underline{i} + (\ddot{y} + 2w\dot{x} - w^2y)\underline{j} \qquad \ldots(2.19)$$

Thus the acceleration of the particle consists of three separate terms:

$$\underline{a} = (d\underline{V}/dt) = (\ddot{x} - 2\omega\dot{y} - \omega^2x)\underline{i} + (\ddot{y} + 2\omega\dot{x} - \omega^2y)\underline{j} \; (2)$$

a) $\ddot{x}\,\underline{i} + \ddot{y}\,\underline{j}$; acceleration with respect to the rotating coordinate system, $\ddot{x}\underline{i} + \ddot{y}\underline{j}$ (i)

b) $2w(-\dot{y}i + \dot{x}j) = 2\underline{w} \wedge \underline{v}_r$; Coriolis acceleration, where $\underline{v}_r = \dot{x}\underline{i} + \dot{y}\underline{j}$ is the relative velocity and $\underline{w} = w\underline{k}$.

$2\omega(-\dot{y}\underline{i} + \dot{x}\underline{j}) = 2\,\underline{\omega} \times \underline{v}_r$ ① (Coriolis acceleration)

$\underline{v}_r = \dot{x}\underline{i} + \dot{y}\underline{j}$ (is the relative velocity) i.e. as if $\underline{i}$ & $\underline{j}$ fixed.

$\underline{\omega} = \omega\underline{k}$ (ii)

3- Theorem: With the hypothesis of theorem -2- the vector $[N'(s) + \kappa(s)\, T(s)]$ is orthogonal to the osculating plane to h at s.

Proof :- It is enough to prove that $N'(s) + \kappa(s)\, T(s)$ is orthogonal to $T(s)$ and to $N(s)$. We have $(N'(s) + \kappa(s)\, T(s)) \cdot T(s) = (N'(s) \cdot T(s)) + \kappa(s) = 0$ by [illegible] above, $(N'(s) + \kappa(s)\, T(s)) \cdot N(s) = (N'(s) \cdot N(s)) + 0 = 0$ $\square$

-b- Definition Suppose that the function $h : E \subseteq \mathbb{R} \to \mathbb{R}^3$ is a three times diffrentiable path-length parametrization of a curve in $\mathbb{R}^3$ with $\kappa(s) \neq 0$ $\forall s \in E$. [i] For each $s \in E$ the unit vector $B(s) = T(s) \times N(s)$, is called the binormal to h at s. [ii] By Theorem -3- there exist a number $\tau(s)$ s.t $N'(s) + \kappa(s)\, T(s) = \tau(s)\, B(s)$.

The number is called the torsion of h at s.

4- Theorem (J.A. Serret, F-J. Frénet). In the notation of Definition -b-, the three vector-valued functions T, N, B on E and the two real valued functions κ and τ on E satisfy the following formulae: [i] $T' = \kappa N$ [ii] $N' = -\kappa T + \tau B$ [iii] $B' = -\tau N$.

Notice that it is a consequence of (ii) of theorem -2- above that $h(E)$ lies in a plane $\Leftrightarrow$ $B'(s) = 0$ $\forall s \in E$. Therefore $h(E)$ is planar $\Leftrightarrow$ the binormal $B(s)$ is constant. It will be seen from the relation $B' = -\tau N$ and ~~by~~ Fig. page 15 that the torsion $\tau(s)$ provides a measure of the rate at which the osculating plane twists about the tangent line at $h(s)$, the twist being clockwise in the direction of orientation when $\tau(s)$ is positive and counterclockwise when $\tau(s)$ is negative.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = \underline{\omega} \times \underline{r} = -\omega y\, \underline{i} + \omega x\, \underline{j}$$

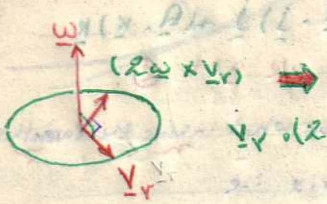
$$\underline{V} = d\underline{r}/dt = \dot{\underline{r}} + (\underline{\omega} \times \underline{r}) \quad (1) \quad \|\underline{\omega}\| = \text{Const.}$$

which means that,

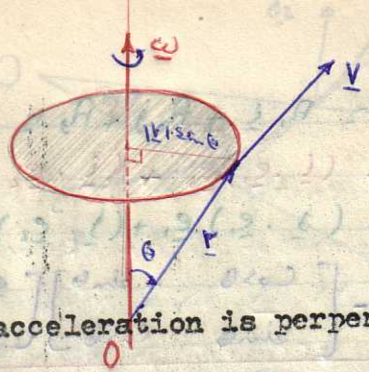
$$\left(\frac{d\square}{dt}\right)_{\text{absolute}} = \left(\frac{d\square}{dt}\right)_{\text{relative}} + \underline{\omega} \times \square \quad (2)$$

(i.e. as if $\underline{i}$ & $\underline{j}$ are fixed)

That is to say, the absolute differential factor is obtained when the differential operator on the R.H.S operates on a vector valued function in a rotating frame.

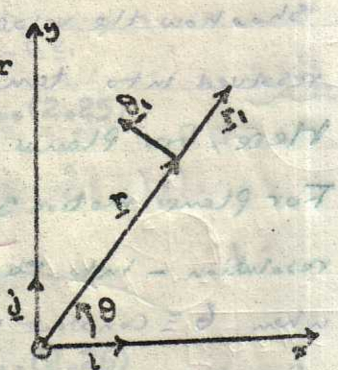


$\underline{v}_r \cdot (2\underline{\omega} \times \underline{v}_r) = 0$



Since $\underline{v}_r \cdot (2\underline{w} \wedge \underline{v}_r) = 0$, the Coriolis acceleration is perpendicular to the relative velocity.

c) $-w^2 x\underline{i} - w^2 y\underline{j} = -w^2\underline{r}$; the centripetal acceleration and is directed radially towards the centre of rotation.



(c) (ii) Polar Coordinates: Let (r,θ) represent the polar coordinates describing the position of a particle at any time t. If $\underline{r}_1$ is a unit vector in the direction of $\underline{r}$ and $\underline{\theta}_1$ is a unit vector perpendicular to $\underline{r}$ and in the direction of increasing θ.

$\partial\underline{r}/\partial r$ is a vector tangent to the curve θ=const. i.e. a vector in the direction of r (increasing r). A unit vector in this direction is thus given by

$$\underline{r}_1 = \frac{\partial \underline{r}}{\partial r} \Big/ \left|\frac{\partial \underline{r}}{\partial r}\right| \qquad (1)$$

Since $\underline{r} = x\underline{i} + y\underline{j} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j}$

Then $\dfrac{\partial \underline{r}}{\partial r} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$, $\left|\dfrac{\partial \underline{r}}{\partial r}\right| = 1$ ...(2.20)

so that $\underline{r}_1 = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} = \underline{e}_1$ ...(2.21)

Similarly, $\dfrac{\partial \underline{r}}{\partial \theta}$ is a vector tangent to the curve r=const. A unit vector in this direction is given by

$$\underline{\theta}_1 = \frac{\partial \underline{r}}{\partial \theta} \Big/ \left|\frac{\partial \underline{r}}{\partial \theta}\right| \qquad (2)$$

Then $\dfrac{\partial \underline{r}}{\partial \theta} = -r\sin\theta\,\underline{i} + r\cos\theta\,\underline{j}$, $\left|\dfrac{\partial \underline{r}}{\partial \theta}\right| = r$ ...(2.22)

so that $\underline{\theta}_1 = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} = \underline{e}_2$ ...(2.23)

$$2\begin{vmatrix} \underline{i} & \underline{j} & k \\ 0 & 0 & \omega \\ \mathring{x} & \mathring{y} & 0 \end{vmatrix} = 2\,\underline{\omega} \times \mathring{\underline{r}}_r =$$

That is to say,

$$\underline{a} = \underbrace{\ddot{\underline{r}}_r}_{\substack{\text{relative}\\ \text{acceleration}\\ \text{i.e. as if } \underline{i},\underline{j} \text{ fixed}}} + \underbrace{2(\underline{\omega} \times \mathring{\underline{r}}_r)}_{\substack{\text{Coriolis}\\ \text{acceleration}}} - \underbrace{\omega^2 \underline{r}}_{\substack{\text{Centripental}\\ \text{acceleration}}} \quad (3)$$

Therefore when (2) operates on $\underline{V}$ ⇒

$$\underline{a} = (d\underline{V}/dt)_{\text{absolute}} = (d\underline{V}/dt)_{\text{relative}} + \underline{\omega} \times \underline{V} =$$

$$\left(\frac{d}{dt}\right)_{\text{relative}} (d\underline{r}/dt) + \underline{\omega} \times (d\underline{r}/dt) =$$

$$(4) \quad \left(\frac{d}{dt}\right)_r (\mathring{\underline{r}} + (\underline{\omega} \times \underline{r})) + \underline{\omega} \times (\mathring{\underline{r}} + (\underline{\omega} \times \underline{r}))$$

$$\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} =$$

$$-\omega^2 x \underline{i} + \omega^2 y \underline{j}.$$

Centrifantal acceleration $=$

$$-\omega^2 \underline{r} =$$

$$-\omega^2 (x\underline{i} + y\underline{j})$$

(iii) Hence

its direction is towards the center of rotation.

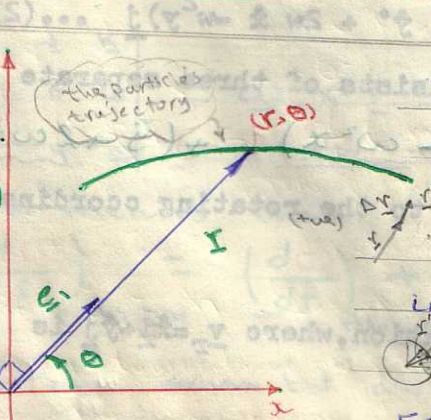
Notice:- $2(\underline{\omega} \times \mathring{\underline{r}}_r) \perp \underline{V}_r = \mathring{\underline{r}}_r$.

(Coriolis acceleration)

From (4) $= \ddot{\underline{r}}_r + \underline{\omega} \times \mathring{\underline{r}}_r + \boxed{\cancel{\dot{\underline{\omega}} \times \underline{r}}} + \underline{\omega} \times \mathring{\underline{r}} + [\underline{\omega} \times (\underline{\omega} \times \underline{r})]$

$$\ddot{\underline{r}}_r + \underline{\omega} \wedge \mathring{\underline{V}}_r + \underline{\omega} \times \mathring{\underline{r}}_r + \boxed{\cancel{(\underline{\omega} \cdot \underline{r})}\,\underline{\omega}} - (\underline{\omega} \cdot \underline{\omega})\,\underline{r} = \underline{a}$$

Where r and θ are Called Polar Functions



For fixed $\underline{i}$ & $\underline{j}$ ⇒ $\boxed{\underline{e}_1 = \frac{\partial \underline{r}}{\partial r} \Big/ \left|\frac{\partial \underline{r}}{\partial r}\right|}$

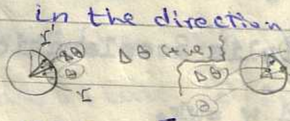
– in the direction of increase of r for

$\partial \underline{r}$ is (+ve) or (−ve) ⇒ $\partial r = |\underline{r}'| - |\underline{r}| =$ (+ve) or (−ve)

$$\boxed{\underline{e}_2 = \frac{\partial \underline{r}}{\partial \theta} \Big/ \left|\frac{\partial \underline{r}}{\partial \theta}\right|}$$

in the direction of increase of θ for

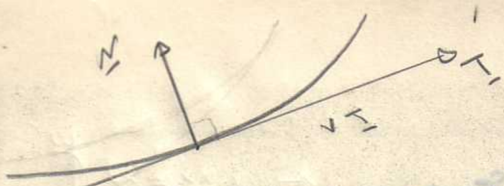
$\delta s = \delta\theta\, r_1$

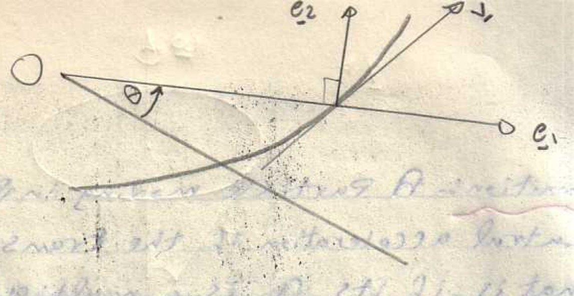


Δθ (−ve) ; $\partial \underline{r} = \underline{r}' - \underline{r}$ is (+ve) or (−ve)

① $$\begin{bmatrix} \underline{e}_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix}$$ which is the rotation matrix

where $\underline{e}_1$ & $\underline{e}_2$ are rotating. From $\{\underline{i}$ & $\underline{j}\}$ to $\{\underline{e}_1$ and $\underline{e}_2\}$





Solving (2.21) and (2.23) we get:

$$\underline{i} = \cos\theta\ \underline{r}_1 - \sin\theta\ \underline{\theta}_1 \quad \& \quad \underline{j} = \sin\theta\ \underline{r}_1 + \cos\theta\ \underline{\theta}_1 \qquad \ldots(2.24)$$

Since $\dot{\underline{r}}_1 = \frac{d\underline{r}_1}{dt} = \overset{=0}{\boxed{\frac{\partial \underline{r}_1}{\partial r}}}\frac{dr}{dt} + \frac{\partial \underline{r}_1}{\partial\theta}\frac{d\theta}{dt} \Rightarrow \dot{\underline{e}}_1 = \dot\theta\,\underline{e}_2$

and $\dot{\underline{\theta}}_1 = \frac{d\underline{\theta}_1}{dt} = \frac{\partial\underline{\theta}_1}{\partial r}\frac{dr}{dt} + \frac{\partial\underline{\theta}_1}{\partial\theta}\frac{d\theta}{dt} \Rightarrow \dot{\underline{e}}_2 = -\dot\theta\,\underline{e}_1$

Then $\dot{\underline{r}}_1 = \dot\theta\ \underline{\theta}_1 \quad \& \quad \dot{\underline{\theta}}_1 = -\dot\theta\ \underline{r}_1 \qquad \ldots(2.25)$

using Eqs.(2.20) and (2.22).

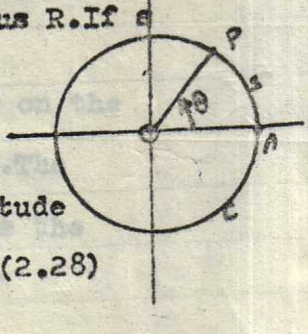
Since $\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\underline{r}_1) = \frac{dr}{dt}\underline{r}_1 + r\frac{d\underline{r}_1}{dt}$

Then $\underline{v} = \dot r\ \overset{\underline{e}_1}{\underline{r}_1} + r\ \dot\theta\ \overset{\underline{e}_2}{\underline{\theta}_1} \qquad \ldots(2.26)$

Also $\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(\dot r\underline{r}_1 + r\dot\theta\underline{\theta}_1)$

Then $\underline{a} = (\ddot r - r\ \dot\theta^2)\overset{\underline{e}_1}{\underline{r}_1} + (r\ \ddot\theta + 2\ \dot r\ \dot\theta)\ \overset{\underline{e}_2}{\underline{\theta}_1} \qquad \ldots(2.27)$

Suppose a particle p moves on a circle C of radius R. If s is the arc length, measured along C from A to p and θ is the corresponding angle subtended at the centre O, then $s = R\theta$. Thus the velocity is tangential of magnitude



$$v_T = R\dot\theta \qquad \ldots(2.28)$$

The radial acceleration is

$$a_r = -R\dot\theta^2 = -v_T^2/R \qquad \ldots(2.29)$$

The tangential acceleration is

$$a_T = R\ddot\theta \qquad \ldots(2.30)$$

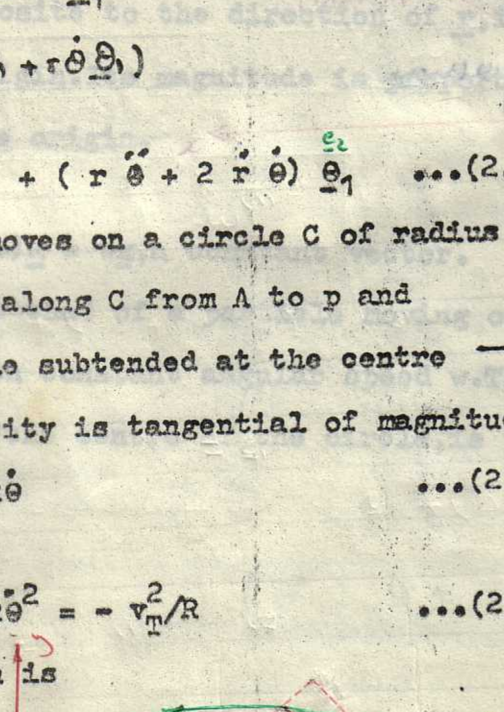
$0<m<1$ by mean value theorem

$dA = \frac{1}{2}\,(\overset{\text{base}}{r(t+\Delta t)-r(t)})(\overset{\text{height}}{r(t+m\Delta t)})$

$\Rightarrow \frac{dA}{dt} = \frac{1}{2}\lim_{\delta t\to 0}\frac{\delta S}{\delta t}\,r(t+m\Delta t)$

$\delta S = r(t)\,\delta\theta \Rightarrow$

$\frac{1}{2}\lim_{\delta t\to 0}[r(t+m\Delta t)]^2\frac{\delta\theta}{\delta t} = \boxed{\frac{1}{2}(r(t))^2\frac{d\theta}{dt} = \frac{dA}{dt} = \frac{1}{2}r^2\dot\theta}$



Since $\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k} = (\underline{A}\cdot\underline{i})\,\underline{i} + (\underline{A}\cdot\underline{j})\,\underline{j} + (\underline{A}\cdot\underline{k})\,\underline{k}$

$\Rightarrow \quad \underline{i} = (\underline{i}\cdot\underline{e}_1)\,\underline{e}_1 + (\underline{i}\cdot\underline{e}_2)\,\underline{e}_2$ $\Rightarrow$ { from ① we get ②

$\underline{j} = (\underline{j}\cdot\underline{e}_1)\,\underline{e}_1 + (\underline{j}\cdot\underline{e}_2)\,\underline{e}_2$ Solving for what inside the parentheses }

② $$\begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix}$$ rotation matrix i.e (from $\underline{e}_1$ and $\underline{e}_2$ to the fixed $\underline{i}$ & $\underline{j}$)

$$\boxed{\begin{array}{l} \underline{v} = \dot{r}\,\underline{e}_1 + r\dot{\theta}\,\underline{e}_2 \quad ; \quad \underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_1 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e}_2 \\ \text{When } \ddot{\theta} = 0 \Rightarrow \underline{a} = (\underset{(1)}{\ddot{r}} - \underset{(2)}{r\dot{\theta}^2})\,\underline{e}_1 + \underset{(3)}{2\dot{r}\dot{\theta}\,\underline{e}_2} \end{array}}$$

$\Rightarrow$ we have three accelerations. Notice for page (15) [1] in $\mathbb{R}^3$

Show how the velocity and the acceleration of a particle can be resolved into tangential and normal components.

Here, for planar motion $\underline{v}$ and $\underline{a}$ are given by polar functions.

For planar motion given by polar functions there is another useful resolution - into radial and transverse components.

when $\dot{\theta} = \text{constant} \Rightarrow \dot{\theta} = \omega$ which is called the angular speed.

for the above (1) $\ddot{r}$, gives the acceleration for pure radial motion.

(2) $-r\dot{\theta}^2\underline{e}_1$ stands for the centripetal acceleration.

(3) $2\dot{r}\dot{\theta}\,\underline{e}_2$ is known as the Coriolis acceleration, this term is due partly to the change in direction of the radial component of $\underline{v}$, and partly to the fact that, as r changes the transverse component of $\underline{v}$ changes; even if $\omega = \frac{d\theta}{dt}$ is constant.

Since $\underline{F} = m\underline{a} = F_r\,\underline{e}_1 + F_\theta\,\underline{e}_2$ The motion of the particle is then governed by the two differential equations

$F_r = m(\ddot{r} - r\dot{\theta}^2)$; $F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

$\Rightarrow \quad rF_\theta = \frac{d}{dt}\left(mr^2\left(\frac{d\theta}{dt}\right)\right)$ stating that the torque applied to the particle equals the time rate of change of its angular momentum. If $F_\theta = 0 \Rightarrow$

$mr^2\dot{\theta} = h$. In other words, if the force is always directed radially towards or away from the origin (a "central force field"), then the angular momentum of the particle will be constant.

This immediately implies Kepler's second law of planetary motion, that the radius vector in a central force field sweeps over area at a constant rate i.e $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$.

Ex.2.5: A particle moves so that its position vector is given by $\underline{r} = \cos wt\, \underline{i} + \sin wt\, \underline{j}$, where w is a constant. Show that the velocity $\underline{v}$ of the particle is perpendicular to $\underline{r}$ and the acceleration $\underline{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin. Hence prove that $\underline{r} \wedge \underline{v} =$ a constant vector.

Since $\underline{v} = \frac{d\underline{r}}{dt} = -\omega \sin \omega t\, \underline{i} + \omega \cos \omega t\, \underline{j}$

Then $\underline{r} \cdot \underline{v} = (\cos \omega t\, \underline{i} + \sin \omega t\, \underline{j}) \cdot (-\omega \sin \omega t\, \underline{i} + \omega \cos \omega t\, \underline{j}) = 0$

thus $\underline{r}$ and $\underline{v}$ are perpendicular.

$$\underline{a} = \frac{d\underline{v}}{dt} = -\omega^2 \cos \omega t\, \underline{i} - \omega^2 \sin \omega t\, \underline{j} = -\omega^2 \underline{r}$$

Thus the acceleration is opposite to the direction of $\underline{r}$, i.e. it is directed toward the origin. Its magnitude is proportional to $|\underline{r}|$, the distance from the origin.

$$\underline{r} \wedge \underline{v} = w(\cos^2 wt + \sin^2 wt)\underline{k} = w\underline{k}, \text{ a constant vector.}$$

Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed w. The acceleration, directed toward the centre of the circle, is the centripetal acceleration.

Definition:- A particle moving in $\mathbb{R}^2$ is said to experience a central acceleration if the transverse component of $\underline{a}$ is ziro, that is, if its $\underline{a}$ is a multiple of $\underline{r}$.

A central acceleration may be pictured as drawn from the particle towards or away from the origin.

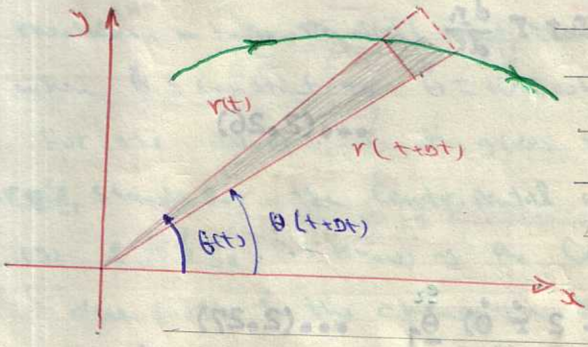
Theorem:- If a twice-differentiable function $\underline{r} : D \subseteq \mathbb{R} \to \mathbb{R}^2$ describes a motion in $\mathbb{R}^2$ under a central acceleration, then there is a constant h s.t $r^2\dot\theta = h \quad \forall\, \underline{r} \in \mathbb{R}^2$

Proof:- if the function $\underline{r}$ describes a central acceleration,

$\Rightarrow 2\dot r\dot\theta + r\ddot\theta \equiv 0$, ~~an~~ i.e. is the ziro function on D.

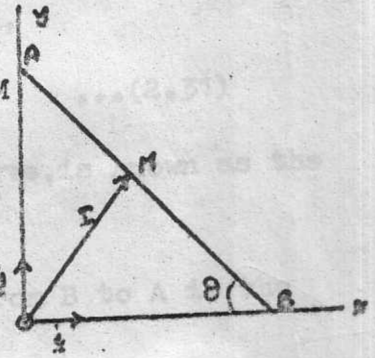
since $(r(2\dot r\dot\theta + r\ddot\theta)) = d(r^2\dot\theta)/dt \equiv 0$ ■

[look page 188) of (Vector Calculus)]



Ex.2.6: A ladder AB of length a rests against a vertical wall OA. The foot B of the ladder is pulled away with constant speed v_0. (a) Show that the midpoint of the ladder describes the arc of a circle of radius a/2 with centre at O. (b) Find the velocity and speed of the midpoint of the ladder at the instant where B is distant b<a from the wall.

Let $\underline{r}$ be the position vector of the midpoint M of AB. If the angle OBA = θ, we have



$$\underline{OB} = a\cos\theta\,\underline{i}\ ,\ \underline{OA} = a\sin\theta\,\underline{j}\ ,\ \underline{AB} = a\cos\theta\,\underline{i} - a\sin\theta\,\underline{j}$$

Then
$$\underline{r} = \underline{OA} + \underline{AM} = \underline{OA} + \tfrac{1}{2}\underline{AB}$$
$$= \tfrac{1}{2}a(\cos\theta\,\underline{i} + \sin\theta\,\underline{j})$$

Thus $|\underline{r}| = a/2$, which is a circle of radius a/2 with centre at O.

The velocity of the midpoint M is
$$\frac{d\underline{r}}{dt} = \tfrac{1}{2}a(-\sin\theta\,\dot{\theta}\,\underline{i} + \cos\theta\,\dot{\theta}\,\underline{j})$$

The velocity of the foot B of the ladder is
$$v_0\,\underline{i} = \frac{d}{dt}(\underline{OB}) = -a\sin\theta\,\dot{\theta}\,\underline{i}$$

At the instant where B is distant b from the wall we have
$$\sin\theta = \sqrt{a^2-b^2}\,/a\quad ,\ \dot{\theta} = -v_0/\sqrt{a^2-b^2}\ ,\ \cos\theta = b/a$$

Thus the required velocity of M at this instant is
$$\frac{d\underline{r}}{dt} = \tfrac{1}{2}v_0\,(\underline{i} - b\underline{j}/\sqrt{a^2-b^2})$$

and its speed is $av_0/2\sqrt{a^2-b^2}$.

a Central force field $\underline{F} = f(r)(\underline{r}/r)$

$$\frac{d\underline{r}}{ds}\cdot\frac{ds}{dt} = \underline{v} = \frac{d\underline{r}}{dt}$$ => equivalent to a tangent vector =>

α
3

acceleration $= \dfrac{d\underline{v}}{dt} =$ equivalent to the Principle normal

$$\frac{d\underline{v}}{dt} = \frac{d(d\underline{r}/ds)}{dt}\cdot\frac{ds}{dt} + \frac{d\underline{r}}{ds}\frac{d^2s}{dt^2} = \frac{d(d\underline{r}/ds)}{ds}\cdot\frac{ds}{dt}\cdot\frac{ds}{dt}$$

$$\Rightarrow \underline{a} = \frac{d\underline{v}}{dt} = \left(\frac{ds}{dt}\right)^2\cdot\frac{d^2\underline{r}}{ds^2} + \frac{d^2s}{dt^2}\cdot\frac{d\underline{r}}{ds}$$

$$\underline{r}\times\underline{v} = r\underline{e}_1 \times (\dot{r}\underline{e}_1 + r\dot{\theta}\underline{e}_2) =$$

$$\begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix} = r \times r\dot{\theta}\ \underline{e}_3 = r^2\dot{\theta}\,\underline{e}_3 \Rightarrow$$

$$\boxed{\underline{r}\times\underline{v} = r^2\dot{\theta}\,\underline{e}_3}$$ where $r^2\dot{\theta}\,\underline{e}_3 = \underline{h} = h\underline{e}_3$

$$\Rightarrow \boxed{\underline{r}\times\underline{v} = \underline{h}}$$

$F(r)$

3. Central forces and Orbital motion :-

take a force $\underline{F} = f(r)(\underline{r}/r)$ ① $\underline{F} = f(r)\,\underline{e}_1$

and one of the components of $\underline{a}$ is

$$\underline{a}_1 = -r\dot{\theta}^2\,\underline{e}_1 \quad ②$$

from ① + ② $\underset{\text{(receeding from O)}}{f(r) > 0}$, $\underset{\text{(towards O)}}{f(r) < 0}$

thus from a magnitude point of view when $\dot{\theta} = \text{const}$

$$\underline{r} \times \underline{F} = \underline{r} \times [\,f(r)\,\underline{r}\,]/r$$

multiplying ① by $\underline{r}$ $\Rightarrow$ $\underline{r} \times f(r)\,\underline{e}_1$

$$\frac{f(r)}{r}(\underline{r} \times \underline{r}) = 0$$

$$\underline{F} = m\frac{d\underline{v}}{dt} \Rightarrow \underline{r} \times \underline{F} = 0 = \underline{r} \times m\left(\frac{d\underline{v}}{dt}\right) =$$

$$m\left(\underline{r} \times \frac{d\underline{v}}{dt}\right) = 0$$ a is in the direction of motion

$$\boxed{\underline{r} \times \underline{v} = \underline{h}} \quad ③ \Rightarrow \underline{r}\cdot[\underline{r} \times \underline{v}] = \underline{r}\cdot\underline{h} = 0 \quad \boxed{4}$$

$\underline{h} \equiv$ angular momentum for central forces

Notice we have reached this from $\frac{d}{dt}(\underline{r} \times \underline{v}) = \overset{=0}{\boxed{\underline{r} \times \frac{d\underline{v}}{dt}}} + \overset{=0}{\boxed{\frac{d\underline{r}}{dt} \times \underline{v}}}$
$= 0$ (which is a special case for a central force field.)

$\underline{r}\cdot\underline{h} = 0$ is an equation of a plane

where $\underline{h}$ normal to the plane.

i.e. ~~all the~~ motion takes place in a plane.

the angular momentum $m\underline{h}$ is a constant (or conserved)

$\underline{F}(r) = (f(r), 0) = m\,\underline{a} \Rightarrow \underline{F}(r)\cdot\underline{V} = \dot{r}\, f(r) \Rightarrow$

$\underline{F}(r)\cdot\underline{V} = m\,\underline{a}\cdot\underline{V}\,dt - f(r)\frac{dr}{dt}\,dt = 0 \Rightarrow$ since $\underline{a}\cdot V = \frac{1}{2}\frac{d\|\underline{V}\|^2}{dt} \Rightarrow$

$\frac{1}{2}m\int \frac{d\|\underline{V}\|^2}{dt}\,dt - \int f(r)\frac{dr}{dt}\,dt = \text{Const.}$

The Equation of Motion :-

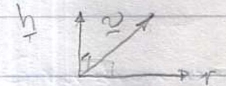
$$m(\ddot{r} - r\dot{\theta}^2)\,\underline{e}_1 = f(r)\,\underline{e}_1 \quad ⑤$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \quad ⑥$$

which are the components of the two accelerations in $\underline{e}_1$, $\underline{e}_2$ directions

$$m\,\frac{d(r^2\dot{\theta})}{dt} = 0 \Rightarrow \frac{d(r^2\dot{\theta})}{dt} = 0,$$

$$\Rightarrow r^2\dot{\theta} = \text{constant} = h = |\underline{r}\times\underline{v}|$$



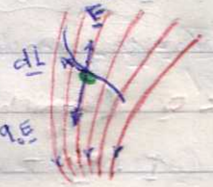
$\Rightarrow \underline{r}\times\underline{v} = 2\,\text{areal velocity} = \text{Costant.}$

$$\underline{r}\times\underline{v} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix} = r^2\dot{\theta}\,\underline{e}_3 = r^2\dot{\theta}\,\underline{e}_3 \quad \Big\}\ \Delta V = \frac{\Delta W}{q_0}$$

$\underline{F}\cdot d\underline{r} = -d(V) \rightarrow$ potential (→ve) will be explained later.

the work done by an external agent to keep (q_0) from accelerating

$$\underline{V} = -\int \underline{F}\cdot d\underline{r} \quad \rightarrow ⑧$$



$$\Delta W = \underline{F}\cdot\Delta\underline{r} \Rightarrow \frac{dW}{q_0} = dV = -\underline{E}\cdot d\underline{r}$$

$$= -\int f(r)\,\underline{e}_r\cdot d\underline{r} = -\int f(r)\,dr$$

(* Go below)

⑤ × $\dot{r}$; ⑥ × $r\dot{\theta}$ and adding them,

$$m(\dot{r}\ddot{r} - r\dot{r}\dot{\theta}^2)\,\cancel{\underline{e}_1} = \dot{r}\,f(r)\,\cancel{\underline{e}_1} \quad [1]$$

$$m(2r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta}) = 0 \quad [2] \Rightarrow [1] + [2]$$

i.e. for a Central force field.

$$m\,\underline{a}\cdot\underline{V} = [1] + [2] = (f(r), 0)\cdot\underline{V} = \underline{F}\cdot\underline{V}$$

which is a
Power Like
equation

$\Rightarrow$ multiplying by $(dt) \Rightarrow$ ②

from ① $\Rightarrow \frac{m}{2}\left(\frac{d}{dt}(\dot{r}^2+r^2\dot{\theta}^2)\right)\underline{dt} - f(r)\left(\frac{dr}{dt}\right)\underline{dt} = 0$

$\Rightarrow \frac{m}{2}\int d(\dot{r}^2+r^2\dot{\theta}^2) - \int f(r)\,dr = \text{Constant}$ ②

$\frac{m}{2}[2\dot{r}\ddot{r}+2r^2\dot{\theta}\ddot{\theta}+2r\dot{r}\dot{\theta}^2] = f(r)\,\dot{r}$ ①

$\Rightarrow$ equivalent to $\frac{m}{2}\frac{d}{dt}\underbrace{\boxed{(\dot{r}^2+r^2\dot{\theta}^2)}}_{=v^2} = \frac{d}{dt}\int f(r)\,dr$

(kinetic energy) (potential energy)

$\Downarrow$

$\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) - \int f(r)\,dr = \text{constant}$

$\Rightarrow$ from ② $\frac{m}{2}(\dot{r}^2+r^2\dot{\theta}^2) - \int f(r)\,dr = \text{Const.}$

$$K + V = \text{Constant.} \qquad (9)$$

since $\underline{V} = \dot{r}\underline{e}_1 + r\dot{\theta}\underline{e}_2 \Rightarrow |\underline{V}|^2 = (\dot{r}^2+r^2\dot{\theta}^2)^{2/2}$

$V(x,y,z) \Rightarrow$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \qquad (10)$$

$$dV = \left(\frac{\partial V}{\partial x}\underline{i} + \frac{\partial V}{\partial y}\underline{j} + \frac{\partial V}{\partial z}\underline{k}\right)\cdot\left(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}\right)$$

$$\Rightarrow dV = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)V\cdot d\underline{r}$$

$$\Rightarrow\ dV = \nabla V\cdot d\underline{r}$$

i.e a differential coefficient put its influence on V

$\nabla \equiv$ vector differential operator (nabla)

$$\nabla \equiv \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$\underline{F}(r) = (f(r), 0) = m(\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$, $\underline{v} = (\dot{r}, r\dot{\theta}) \Rightarrow$

$\underline{F}(r)\cdot\underline{v} = \dot{r}f(r) = m(\dot{r}\ddot{r} - \underline{r\dot{r}\dot{\theta}^2} + r^2\dot{\theta}\ddot{\theta} + \underline{2r\dot{r}\dot{\theta}^2}) =$

$\boxed{\frac{m}{2}\frac{d\|\underline{v}\|^2}{dt} = \dot{r}f(r)}$ ① $\frac{m}{2}\times 2(\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta}) = \frac{m}{2}\frac{d(\dot{r}^2+r^2\dot{\theta}^2)}{dt} =$

Now got any $\underline{V}$

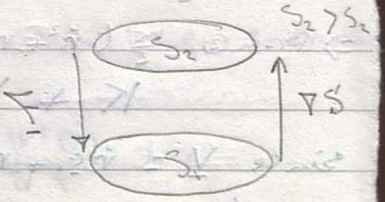
∇S Gradient of a Scalar S (grad S)

$\nabla . \underline{V}$ Divergence of a vector $\underline{V}$ (div $\underline{V}$)

$\nabla \times \underline{V}$ Curl of a vector (Curl $\underline{V}$)

Surface on it potential is constant

$\Rightarrow$ got any two surfaces for a V



∇S rate of increase of S which explaines

the -ve sign in (8)

Gradien of a Scalar S $\nabla S = \frac{\partial S}{\partial n} \underline{n}$

where $\underline{n}$ unit vector is $\perp$ to the two surfaces

$$\nabla S \cdot d\underline{r} = \frac{\partial S}{\partial n} \underline{n} \cdot d\underline{r} = \frac{\partial S}{\partial n} dn \; ; \; \underline{n} = \frac{\nabla S}{|\nabla S|}$$

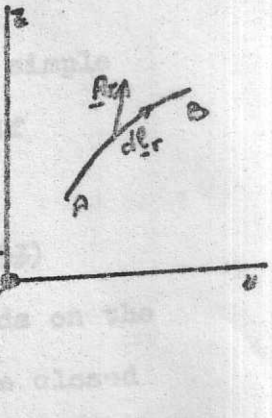
$$= \left(\frac{\partial S}{\partial x} \underline{i} + \frac{\partial S}{\partial y} \underline{j} + \frac{\partial S}{\partial z} \underline{k} \right) \cdot \underbrace{(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k})}_{= d\underline{r}}$$

ex. ① find the normal unit vector to surface $S = xy^2z^3$

② $S = (x^2+y^2) - 4z$ at $(1,2,3)$

① $\nabla S = 2x\underline{i} + 2y\underline{j} - 4\underline{k} = 2\underline{i} + 4\underline{j} - 4\underline{k}$

$\nabla S = \sqrt{4+16+16} = 6 \Rightarrow \underline{n} = \frac{2\underline{i} + 4\underline{j} - 4\underline{k}}{6}$



(4) 2.4 The Line Integral:

Let $\underline{A}$ be a vector field in space and AB a curve described in the sense A to B. Let the continuous curve AB be subdivided into infinitesimal vector elements $d\underline{l}_r$ (r=1,2,...n). Take the scalar products $\underline{A}_r \cdot d\underline{l}_r$, where $\underline{A}_r$ are the values that the vector field $\underline{A}$ takes at the junction points of the vectors $d\underline{l}_r$. The sum of these scalar products, that is,

$$\circledast \quad \sum_{A}^{B} \underline{A}_r \cdot d\underline{l}_r = \int_A^B \underline{A} \cdot d\underline{l} \qquad \ldots(2.31)$$

summed up along the entire length of the curve, is known as the line integral of A along the curve AB.

It is obvious that the line integral from B to A is the negative of that from A to B.

Let $\underline{A} = \underline{A}(x,y,z) = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$ be a vector function of position (vector field). The integral of the tangential component of $\underline{A}$ along the curve from A to B is written as:

$$\int_A^B \underline{A} \cdot d\underline{l} = \int_A^B (A_1 dx + A_2 dy + A_3 dz) \qquad \ldots(2.32)$$

If $\underline{F}$ represents the force on a moving particle, then the line integral of $\underline{F}$ over the path described by the particle is the work done by the force.

$$\circledast \quad \boxed{\lim_{\substack{n\to\infty \\ |\Delta \underline{R}_i|\to 0}} \sum_{i=1}^{n=\infty} \underline{F}_i \cdot \Delta \underline{R}_i = \int_C \underline{F} \cdot d\underline{R}}$$

Why

Limit ⊛

exist

doesn't exist

finite

vary

$$\frac{d\underline{R}_i}{dt}\,dt \simeq d\underline{R}_i$$

Provided the magnitude of $|\Delta \underline{R}|$ is sufficiently small s.t the sum of these along C describes well (exactly) the length of the Curve C, the we are confronted with the second problem which is the extent to which $\underline{F}$ changes its direction along each line segment $|\Delta \underline{R}|$ the is an element of the infinite subdivisions i.e as $n\to\infty$. (that affects the total sum)

Now, having the 1st problem solved, to solve the 2nd we think of the subdivision $|\Delta \underline{R}| \to 0$ as $n\to\infty$ s.t along this infinitesimal line segment the field along this line is not given the chance to change its direction, hence for each partition

$\underline{F}_i \cdot \Delta \underline{R}_i$ is finite ⇒ the seq. of partial sums as $|\Delta \underline{R}_i| \to 0$ and $n\to\infty$; in addition to the fact that at each point on the curve $\underline{F}_i \cdot \Delta \underline{R}_i$ is finite ⇒ $\forall i$ $\underline{F}_i \cdot d\underline{R}_i$ is a continuous function, hence we can replace the summation notation by the integral operator, putting in mind that the integral exist and is equivalent to the L.H.S of ⊛

If C is a closed curve(which we shall suppose is a simple closed curve,i.e. a curve which does not intersect itself anywhere)then the integral is often denoted by

$$\oint_c \underline{A}\cdot d\underline{r} = \oint_c (A_1 dx + A_2 dy + A_3 dz) \qquad \ldots(2.33)$$

In general,a line integral has a value which depends on the path.If the line integral is independent of the path,the closed line integràl vanishes identically,a property which will be explained in III.

Ex.2.7:If $A = (3x^2-6yz)i + (2y+3xz)j + (1-4xyz^2)k$,evaluate $\int_c \underline{A}\cdot d\underline{r}$ from (0,0,0)to(1,1,1) along the following paths:

a)$x=t, y=t^2, z=t^3$,

b)the straight lines from (0,0,0)to(0,0,1),then to(0,1,1) and then to (1,1,1),

c)the straight line joining(0,0,0) and(1,1,1).

$$\int_c \underline{A}\cdot d\underline{r} = \int_c \{(3x^2-6yz)\underline{i} + (2y+3xz)\underline{j} + (1-4xyz^2)\underline{k}\}\cdot(dx\underline{i} + dy\underline{j} + dz\underline{k})$$

$$= \int_c [(3x^2-6yz)dx + (2y+3xz)dy + (1-4xyz^2)dz]$$

If $x=t, y=t^2, z=t^3$, points (0,0,0) and (1,1,1) correspond to t=0 and t=1 respectively.Then

$$\int_c \underline{A}\cdot d\underline{r} = \int_{t=0}^{1} [(3t^2-6t^5)dt + (2t^2+3t^4)2t\,dt + (1-4t^9).3t^2dt]$$

$$= 2$$

[illegible] Along the straight line from (0,0,0) to (0,0,1), $x=0, y=0, dx=0, dy=0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1}\left[(1-4(0)(0)(z^2))\,dz\right]=\int_{z=0}^{1}dz=1$$

Along the straight line from (0,0,1) to (0,1,1), $x=0, z=1, dx=0, dz=0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1}2y\,dy=1$$

Along the straight line from (0,1,1) to (1,1,1), $y=1, z=1, dy=0, dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1}(3x^2-6)\,dx=-5$$

Adding

$$\int_C \underset{\sim}{A}\cdot d\underline{r}=1+1-5=-3$$

[illegible] Along the straight line joining (0,0,0) and (1,1,1) we have $x=t$, $y=t, z=t$. Then since $dx=dy=dz=dt$,

$$\int_C A\cdot d\underline{r}=\int_{t=0}^{1}\left[(3t^2-6t^2)dt+(2t+3t^2)dt+(1-4t^4)dt\right]$$

$$=6/5$$

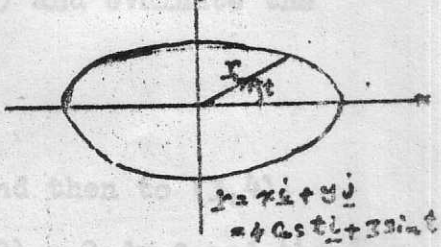
Note that in this case the value of the integral depends on the particular path.

- Ex.2.8:Find the work done in moving a particle once around an ellipse C in the xy plane,if the ellipse has centre at the origin with semi-major and semi-minor axes 4 and 3 respectively,and if the force field is given by

$$\underline{F} = (3x-4y+2z)\underline{i}+(4x+2y-3z^2)\underline{j}+(2xz-4y^2+z^3)\underline{k}$$

In the plane $z=0$, $\underline{F}=(3x-4y)\underline{i}+(4x+2y)\underline{j}-4y^2\underline{k}$ and $d\underline{r}=dx\underline{i}+dy\underline{j}$ so that the work done is

$$\oint_C \underline{F}.d\underline{r} = \int_C \{(3x-4y)dx+(4x+2y)dy\}$$


y
r
t
$\underline{r} = x\underline{i}+y\underline{j}$
$= 4\cos t\underline{i}+3\sin t\underline{j}$

Choose the parametric equations of the ellipse as $x=4\cos t$, $y=3\sin t$,where t varies from 0 to 2π.Then the line integral equals

$$\int_0^{2\pi}(48-30\sin t\cos t)dt = (48t-15\sin^2 t)\Big|_0^{2\pi}$$

$$= 96\pi$$

In traversing C we have chosen the counterclockwise direction. We call this the positive direction,or say that C has been traversed in the positive sense.If C were traversed in the clockwise(negative) direction the value of the integral would be -96π.

Ex.2.9: Evaluate $\int_C y\,ds$ along the curve C given by $y=2\sqrt{x}$ from $x=3$ to $x=24$.

Since $ds=\sqrt{dx^2+dy^2}=\sqrt{1+(y')^2}\,dx=\sqrt{1+1/x}\,dx$, we have

$$\int_C y\,ds=\int_3^{24} 2\sqrt{x}\,\sqrt{1+1/x}\,dx=2\int_3^{24}\sqrt{x+1}\,dx=\frac{4}{3}(x+1)^{3/2}\Big|_3^{24}=156$$

Ex.2.10: Prove that $\int_{1,2}^{3,4}(6xy^2-y^3)\,dx+(6x^2y-3xy^2)\,dy$ is independent of the path joining (1,2) and (3,4) and evaluate the integral.

Choose two paths:

i) Straight lines joining (1,2) to (3,2) and then to (3,4).

Along the straight line from (1,2) to (3,2), $y=2$, $dy=0$ and the line integral equals

$$\int_1^3 (24x-8)\,dx=80$$

Along the straight line from (3,2) to (3,4), $x=3$, $dx=0$ and the line integral equals

$$\int_2^4 (54y-9y^2)\,dy=156$$

Then the required value=236.

ii) Straight line joining (1,2) and (3,4), its equation is $y=x+1$, $dy=dx$, and the integral becomes

Then the required integral equals

$$\int_1^3 \left[(5x-1)(x+1)^2 dx + 3x(x+1)(x-1)\,dx\right] = \int_1^3 \left[8x^3+9x^2-1\right] dx$$

$$= 236$$

i.e. the line integral is independent of the path.

On the other hand, by inspection we could note that

$$\int_{1,2}^{3,4} \left[(6xy^2-y^3)\,dx + (6x^2y-3xy^2)\,dy\right] =$$

$$\int_{1,2}^{3,4} (6xy^2dx + 6x^2y\,dy) - \int_{1,2}^{3,4} (y^3dx + 3xy^2dy)$$

$$= \int_{1,2}^{3,4} d(3x^2y^2) - \int_{1,2}^{3,4} d(xy^3) = \int_{1,2}^{3,4} d(3x^2y^2 - xy^3) = 236$$

Therefore, if the line integral $\int_C \underline{A}\cdot d\underline{r}$ is independent of the path, we can write

$$\int_A^B \underline{A}\cdot d\underline{r} = \int_A^B d\phi = \phi_B - \phi_A$$

Since

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}\right)\cdot\left(dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k}\right)$$

Then $\underline{A} = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$

We have here an important case in which a vector field is derived from a scalar field, if the line integral is independent of the path. Therefore, it does not necessarily follow conversely that all vector fields can be derived from a scalar field.

Problems

2.1: Prove that $\frac{d}{du}(\underline{A}\wedge\underline{B})=\underline{A}\wedge\frac{d\underline{B}}{du}+\frac{d\underline{A}}{du}\wedge\underline{B}$ where $\underline{A}$ and $\underline{B}$ are differentiable functions of u.

Hence show that $\frac{d}{dt}(\underline{V}\wedge\frac{d\underline{V}}{dt})=\underline{V}\wedge\frac{d^2\underline{V}}{dt^2}$

2.2: If $\underline{A} = x^2\sin y\,\underline{i} + z^2\cos y\,\underline{j} - xy^2\,\underline{k}$, find $d\underline{A}$.

Ans. $(2x\sin y\,dx + x^2\cos y\,dy)\underline{i} + (2z\cos y\,dz - z^2\sin y\,dy)\underline{j}$
$-(y^2dx + 2xy\,dy)\underline{k}$

2.3: If $\underline{A} = x^2\underline{i} - y\underline{j} + xz\underline{k}$, $\underline{B} = y\underline{i} + x\underline{j} - xz\underline{k}$ & $\underline{C} = \underline{i} - y\underline{j} + x^3z\underline{k}$ find $\frac{\partial^2}{\partial x\partial z}(\underline{A}\wedge\underline{B})$ & $d[\underline{A}\cdot(\underline{B}\wedge\underline{C})]$, at the point (1,-1,2).

Ans. $-4\underline{i}+8\underline{j}$ & $8dx$

2.4: If $\underline{R} = x^2y\underline{i} - 2y^2z\underline{j} + xy^2z^2\underline{k}$ find $\left|\frac{\partial^2\underline{R}}{\partial x^2}\wedge\frac{\partial^2\underline{R}}{\partial y^2}\right|$ at the point (2,1,-2).

Ans. $16\sqrt{5}$

2.5: If $\underline{r} = (t^3+2t)\underline{i} - 3e^{-2t}\underline{j} + 2\sin 5t\,\underline{k}$, find $\frac{d\underline{r}}{dt}$ and $\frac{d^2\underline{r}}{dt^2}$ at t=0

Ans. $2\underline{i}+6\underline{j}+10\underline{k}$ & $-12\underline{j}$

2.6: If $\underline{A}$ is a differentiable function of u and $|\underline{A}(u)|=1$, prove that $d\underline{A}/du$ is perpendicular to $\underline{A}$.

2.7: Find the unit tangent $\underline{T}$, the principal normal $\underline{N}$ and the radius of curvature to the space curve $x=t, y=t^2/2, z=t$. Hence find the velocity, the speed, and the acceleration of a particle moving on the above curve.

Ans. $\underline{T} = (\underline{i} + t\underline{j} + \underline{k})/\sqrt{t^2+2}$, $\underline{N} = (-t\underline{i} + 2\underline{j} - t\underline{k})/\sqrt{2t^2+4}$

$R = (t^2+2)^{3/2}/\sqrt{2}$, $\dot{\underline{r}} = \underline{i} + t\underline{j} + \underline{k}$, $\sigma = \sqrt{t^2+2}$

$a_T = t/\sqrt{t^2+2}$, $a_r = \sqrt{2}/\sqrt{t^2+2}$

2.8: If $\underline{r} = \underline{a}\cos wt + \underline{b}\sin wt$, where $\underline{a}$ and $\underline{b}$ are any constant non-collinear vectors and w is a constant scalar, prove that

$$\underline{r} \wedge \dot{\underline{r}} = \omega(\underline{a} \wedge \underline{b}) \quad \text{and} \quad \ddot{\underline{r}} + \omega^2 \underline{r} = 0$$

Interpret the relations

$$\underline{r} \cdot \dot{\underline{r}} = 0 \quad \text{and} \quad \underline{r} \wedge \dot{\underline{r}} = 0$$

2.9: Find the tangential and normal acceleration of a particle which moves on the ellipse $\underline{r} = a\cos wt\,\underline{i} + b\sin wt\,\underline{j}$,

$$\text{Ans. } a_T = \frac{\omega^2(a^2-b^2)\sin\omega t\cos\omega t}{\sqrt{a^2\sin^2\omega t + b^2\cos^2\omega t}}$$

$$a_r = \frac{\omega^2 ab}{\sqrt{a^2\sin^2\omega t + b^2\cos^2\omega t}}$$

2.10: Evaluate $\int_{0,1}^{1,2} [(x^2-y)\,dx + (y^2+x)\,dy]$ a) along a straight line from (0,1) to (1,2), b) straight line from (0,1) to (1,1) and then from (1,1) to (1,2), c) the parabola $x=t, y=t^2+1$.

Ans. a) 5/3, b) 8/3, c) 2.

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III-GRADIENT, DIVERGENCE & CURL

3.1 The Differential Operator ∇ :

The vector(differential) operator ∇(del or nabla), in cartesian coordinates is defined by

$$\nabla \equiv \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z} \qquad \ldots(3.1)$$

It may be applied as a directive differentiator either to a scalar or to a vector function of space, provided that these functions have continuous first partial derivatives(at least). We may formally regard ∇ as a vector which can have either a scalar or a vector product with other vectors.

In vector analysis there are three fundamental operations with ∇, which are of physical interest. If S is a scalar function of space and $\underline{V}$ a vector function of space, these operations are:

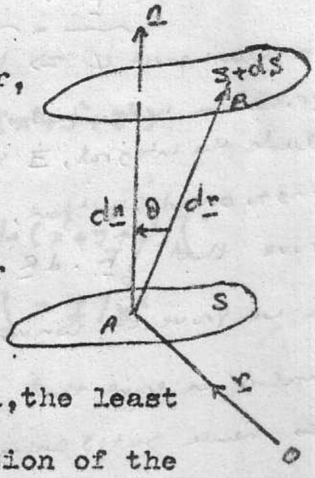
(i) ∇S, where ∇ acts as a differentiator,

(ii) $\nabla . \underline{V}$ and (iii) $\nabla \wedge \underline{V}$, where ∇ is treated as a formal vector.

3.2 The Gradient of a Scalar Field:

Certain physical quantities, such as potential in an electric field due to static charges, can be represented from point to point in space by a scalar point function, S, of the coordinates. The entire scalar field can be mapped out by level surfaces, Go to ① upon each of which the scalar function S has a constant value. In our example, the level surfaces are the equipotential surfaces.

① Which means that for some value of S there exist a level over which S is always constant, this level takes some shape in the region given this shape is then a surface.



Consider two level surface, very close together, and examine a small portion of them in the Neighbourhood of a given point A on the surface characterized by the constant value of S, the scalar function, and the other by S+dS.

If $\underline{r}$ is the radius vector from the origin to A, the least distance between the surfaces will be in the direction of the unit normal vector $\underline{n}$ at A and of length $d\underline{n}$.

The gradient of a scalar function S (or abbreviated grad S) is given by:

$$\text{grad } S = \frac{\partial S}{\partial n}\underline{n} \Rightarrow \nabla S \cdot \underline{n} = \frac{\partial S}{\partial n} \qquad \ldots(3.2).$$

$\nabla \varphi = \frac{\partial \varphi}{\partial s}\underline{u}$ where $\|\underline{u}\| = 1$ and ds is in the direction of $\underline{u}$

It is a vector whose magnitude is the maximum rate of increase of S and its direction is along the normal.

$$\frac{\partial S}{\partial n}\underline{n}\cdot d\underline{r} = (\text{grad } S)\cdot d\underline{r} = (\partial S/\partial n)\cdot d\underline{r} = (\partial S/\partial n)\,\boxed{\overset{= dn}{dr\cos\theta}} = (\partial S/\partial n)\,dn$$

$$\boxed{d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}} = dS = (\partial S/\partial x)dx + (\partial S/\partial y)dy + (\partial S/\partial z)dz$$

$$= (\partial S/\partial x)\underline{i}\cdot dx\underline{i} + (\partial S/\partial y)\underline{j}\cdot dy\underline{j} + (\partial S/\partial z)\underline{k}\cdot dz\underline{k}$$

$$= \left[(\partial S/\partial x)\underline{i} + (\partial S/\partial y)\underline{j} + (\partial S/\partial z)\underline{k}\right]\cdot d\underline{r}$$

Therefore

$$\text{grad } S = (\partial S/\partial x)\underline{i} + (\partial S/\partial y)\underline{j} + (\partial S/\partial z)\underline{k}$$

or

$$\text{grad } S = \nabla S$$

where

$$\nabla S = \frac{\partial S}{\partial x}\underline{i} + \frac{\partial S}{\partial y}\underline{j} + \frac{\partial S}{\partial z}\underline{k} \qquad \ldots(3.3)$$

$\underline{r}(s) = x(s)\,\underline{i} + y(s)\,\underline{j} + z(s)\,\underline{k}$

$f(x,y,z) = x^2+y^2+z^2 = f(\underline{r}(s))$

are the two surfaces parallel?
taking, in general, the whole two surfaces
not parallel => taking a small portion
which include the point at which
we want to describe a property of the
surface => this small portion
should have a parallel portion
on the second surface when the
area of each portion is
shrinked to a point.
Now, the least distance
between the two portions
is conveniently represented
by the normal vector
to the two portions.
Since this distance
tends to zero for the
two surfaces are
close to each other

Scalar field ϕ => isotimic surfaces
=> an arbitrary space curve is drawn
in the region of definition of ϕ
=> generally this curve
continuously intersects the
isotimic surfaces =>
the unit vector tangent to the
curve at some point is $\underline{u}$
=> at this point there is some
vector pointing in the direction
normal to the plane tangent
to the surface at this point
=> this vector is denoted $\nabla\phi$
s.t. $\nabla\phi \cdot \underline{u} = |\nabla\phi|\,|\underline{u}|\cos\theta$
since $|\underline{u}|=1$ & $\nabla\phi\cdot\underline{u}$ is max.
when $\theta = 0$ => $\nabla\phi$ points
in the direction of max. rate
of increase of ϕ => when $\theta = \frac{\pi}{2}$
=> the tangent lies in the
plane tangent to the isotimic
surface at this point
and $\nabla\phi\cdot\underline{u} = 0$ which
means that in this direction
of $\underline{u}$, ϕ doesn't change
in that direction.

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial f}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial f}{\partial z}\cdot\frac{dz}{ds}$$

$$= \nabla f \cdot \frac{d\underline{R}}{ds} \quad \text{where}$$

$f(x,y,z)$ & $\underline{R}(s) = x(s)\,\underline{i} + y(s)\,\underline{j} + z(s)\,\underline{k}$

for the arbitrary curve intersecting the
isotimic surfaces.

$$\boxed{\nabla\phi = \frac{d\phi}{ds}\,\underline{u}}$$

Ex.3.1: Find $\nabla \varnothing$ if (a) $\varnothing = \ln |r|$, (b) $\varnothing = 1/r$

(a) $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. Then $|\underline{r}| = \sqrt{x^2+y^2+z^2}$, $\varphi = \ln|\underline{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)$

$$\nabla\varphi = \frac{1}{2}\nabla \ln(x^2+y^2+z^2)$$

$$= \frac{1}{2}\left\{\underline{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \underline{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \underline{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \frac{1}{2}\left\{\underline{i}\frac{2x}{x^2+y^2+z^2} + \underline{j}\frac{2y}{x^2+y^2+z^2} + \underline{k}\frac{2z}{x^2+y^2+z^2}\right\} = \underline{r}/r^2$$

(b) $\nabla\varphi = \nabla(1/r) = \nabla\left(1/\sqrt{x^2+y^2+z^2}\right) = \nabla\left\{(x^2+y^2+z^2)^{-1/2}\right\}$

$$= \underline{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \underline{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \underline{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \underline{i}\left\{-1/2(x^2+y^2+z^2)^{-3/2}\cdot 2x\right\} + \underline{j}\left\{-1/2(x^2+y^2+z^2)^{-3/2}\cdot 2y\right\} + \underline{k}\left\{-1/2(x^2+y^2+z^2)^{-3/2}\cdot 2z\right\}$$

$$= -\frac{x\underline{i}+y\underline{j}+z\underline{k}}{(x^2+y^2+z^2)^{3/2}} = -\underline{r}/r^3$$

Ex.3.2: Show that $\nabla r^n = n\, r^{n-2}\, \underline{r}$, hence prove that

$$\nabla(\underline{M}\cdot\underline{r})/r^3 = \underline{M}/r^3 - 3(\underline{M}\cdot\underline{r})\underline{r}/r^5$$

where M is a constant vector.

$$\nabla r^n = \nabla\left(\sqrt{x^2+y^2+z^2}\right)^n = \nabla(x^2+y^2+z^2)^{n/2}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}\left\{2x\underline{i} + 2y\underline{j} + 2z\underline{k}\right\} = n(r^2)^{n/2-1}\underline{r} = nr^{n-2}\underline{r}$$

let $\underline{M} = M_1\underline{i} + M_2\underline{j} + M_3\underline{k}$, then $\underline{M}\cdot\underline{r} = xM_1 + yM_2 + zM_3$

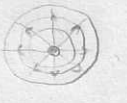
$$\nabla(\underline{M}\cdot\underline{r})/r^3 = \frac{1}{r^3}\nabla(\underline{M}\cdot\underline{r}) + (\underline{M}\cdot\underline{r})\nabla\left(\frac{1}{r^3}\right)$$

$$= \frac{1}{r^3}\left\{M_1\underline{i} + M_2\underline{j} + M_3\underline{k}\right\} + (\underline{M}\cdot\underline{r})\left(-3r^{-5}\,\underline{r}\right)$$

Since $\underline{M}$ is a constant vector & using the above formula.

$$\therefore \nabla(\underline{M}\cdot\underline{r})/r^3 = \underline{M}/r^3 - 3(\underline{M}\cdot\underline{r})/r^5\,\underline{r}$$

$E = -\frac{d\varphi}{dl} = -\nabla\phi \cdot \frac{d\underline{r}}{dl}$



$\Rightarrow -\int E \, dl = \int d\varphi$

— we have a field $\underline{F}$ ⇒ if conservative ⇒ $\int_P^Q \underline{F} \cdot d\underline{R} = \phi(Q) - \phi(P)$

⇒ there must exist ϕ ⇒ to prove some field is conservative ⇒ we suppose the independence of path as a hypothesis ⇒ to evaluate the integral, ∃ some ϕ ⇒ this ϕ must satisfy, according to our definition, $\underline{F} = \nabla\phi$ ⇒ after proving this we prove that $\int_P^Q \underline{F} \cdot d\underline{R} = \phi(Q) - \phi(P)$.

= Then we prove the converse we start by taking $\underline{F}$ as conservative therefore we prove that $\int_P^Q \underline{F} \cdot d\underline{R} = \phi(Q) - \phi(P)$ ~~the~~ hence satisfying the condition of independence of path.

i.e. we want to prove for some field $\underline{F}$

Conservative ⟺ independence of path

and the Conection is the line integral

Think of unification theory using this principle

taking our local space as a conservative field

(b) Equations for the Tangent Plane and the Normal line to a Surface:

Let $\underline{r}$ be the position vector to any point $P(x,y,z)$ on the surface $S(x,y,z)=C$, where C is a constant. Then $d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$ lies in the plane tangent to the surface, at p. But

$$dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = 0$$

or

$$\left(\frac{\partial S}{\partial x}\underline{i} + \frac{\partial S}{\partial y}\underline{j} + \frac{\partial S}{\partial z}\underline{k}\right)\cdot\left(dx\underline{i} + dy\underline{j} + dz\underline{k}\right) = 0$$

i.e.

$$\nabla S\cdot d\underline{r} = 0 \qquad \ldots(3.4)$$

so that ∇S is perpendicular to $d\underline{r}$ and therefore to the surface.

If $\underline{r}$ and $\underline{r}_0$ are the vectors drawn respectively from O to $p(x,y,z)$ and $Q(x_0,y_0,z_0)$ on the tangent plane, the equation of the plane is

$$(\overset{\text{any point on the plane}}{\underline{r}} - \underline{r}_0)\cdot\nabla S|_Q = 0 \qquad \ldots(3.5)$$

since $(\underline{r}-\underline{r}_0)$ is perpendicular to ∇S_Q. In rectangular form this is

$$S_{x|Q}(x-x_0) + S_{y|Q}(y-y_0) + S_{z|Q}(z-z_0) = 0 \qquad \ldots(3.6)$$

where $S_{x|Q} = \frac{\partial S}{\partial x}$ at the point Q, and similar expressions for $S_{y|Q}$ and $S_{z|Q}$

If $\underline{r}$ is the vector drawn from O to any point (x,y,z) on the normal line, then $(\underline{r}-\underline{r}_0)$ is collinear with ∇S_Q and so

$$(\underline{r}-\underline{r}_0)\wedge\nabla S|_Q = 0 \qquad \ldots(3.7)$$

which in rectangular form is

$$\frac{(x-x_0)}{S_{x|Q}} = \frac{(y-y_0)}{S_{y|Q}} = \frac{(z-z_0)}{S_{z|Q}} \qquad \ldots(3.8)$$

Ex.3.3: Prove that the acute angle γ between the z-axis and the normal to the surface $S(x,y,z)=0$ at any point is given by

$$\sec\gamma = \sqrt{S_x^2 + S_y^2 + S_z^2}\ /|S_z|$$

A unit normal to the surface $S(x,y,z)$ is

$$\hat{\underline{n}} = \frac{\nabla S}{|\nabla S|} = \frac{S_x\underline{i} + S_y\underline{j} + S_z\underline{k}}{\sqrt{S_x^2 + S_y^2 + S_z^2}}$$

Since $\cos\gamma = |\underline{k}.\hat{\underline{n}}|$, then

$$\sec\gamma = 1/|\underline{k}.\hat{\underline{n}}| = \sqrt{S_x^2 + S_y^2 + S_z^2}\ /|S_z|$$

Ex.3.4: Find a unit normal to the surface $2x^2+4yz-5z^2=-10$ at the point $p(3,-1,2)$.

A vector normal to the surface is

$$\nabla(2x^2+4yz-5z^2) = 4x\underline{i} + 4z\underline{j} + (4y-10z)\underline{k}$$
$$= 12\underline{i} + 8\underline{j} - 24\underline{k} \qquad \text{at } (3,-1,2)$$

Then a unit normal to the surface at p is

$$\frac{12\underline{i} + 8\underline{j} - 24\underline{k}}{28} = (3\underline{i} + 2\underline{j} - 6\underline{k})/7$$

Another unit normal to the surface at p is

$$-(3\underline{i} + 2\underline{j} - 6\underline{k})/7.$$

Ex.3.5: Find the equations of the tangent plane and the normal line to the surface $x^2+y^2=4z$ at the point $(2,-4,5)$.

Let $S = x^2+y^2-4z=0$

Then $\qquad \nabla S = 2x\underline{i} + 2y\underline{j} - 4\underline{k}$

Thus the equation of the tangent plane at the point (2,-4,5) becomes:

$$2(2)(x-2)+2(-4)(y+4)-4(z-5)=0$$

or $x-2y-z=5$

Also the equation of the normal line at the point (2,-4,5) becomes:

$$\frac{x-2}{4}=\frac{y+4}{-8}=\frac{z-5}{-4}$$

or

$$\frac{x-2}{1}=\frac{y+4}{-2}=\frac{z-5}{-1}$$

(c) The Directional Derivative:

Consider a space curve C and let s be the arclength to a point (x,y,z), measured from some given point on C, then

$$\underline{T} = d\underline{r}/ds \qquad \ldots(3.9)$$

is a unit tangent vector to C at (x,y,z).

If F(x,y,z) is defined at any point on the space curve, we get

$$\frac{dF}{ds}=\frac{\partial F}{\partial x}\frac{dx}{ds}+\frac{\partial F}{\partial y}\frac{dy}{ds}+\frac{\partial F}{\partial z}\frac{dz}{ds}$$

$$=\left(\frac{\partial F}{\partial x}\underline{i}+\frac{\partial F}{\partial y}\underline{j}+\frac{\partial F}{\partial z}\underline{k}\right)\cdot\left(\frac{dx}{ds}\underline{i}+\frac{dy}{ds}\underline{j}+\frac{dz}{ds}\underline{k}\right)$$

or

$$dF/ds = \nabla F\cdot\underline{T} \qquad \ldots(3.10)$$

We often call dF/ds the directional derivative of F at (x,y,z) along C.

Ex.3.6:Find the directional derivative of $F=2xy-z^2$ at(2,-1,1) in the direction toward (3,1,-1).In what direction is the directional derivative a maximum? What is the value of this maximum?

Let $\underline{T}$ be the unit vector at (2,-1,1) in the direction toward (3,1,-1).Then

$$T = (\underline{i} + 2\underline{j} - 2\underline{k})/3$$

The directional derivative

$$\begin{aligned}\frac{dF}{ds} &= \nabla F\cdot T \\ &= (2y\underline{i} + 2x\underline{j} - 2z\underline{k})\cdot(\underline{i} + 2\underline{j} - 2\underline{k})/3 \\ &= (2y + 4x + 4z)/3 \qquad = 10/3 \text{ at}(2,-1,1).\end{aligned}$$

The directional derivative is maximum along the normal to the surface $F= 2xy-z^2$,i.e. along

$$\begin{aligned}\hat{n} &= \nabla F/|\nabla F| = \frac{2y\underline{i} + 2x\underline{j} - 2z\underline{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\ &= (-\underline{i} + 2\underline{j} - \underline{k})/\sqrt{6} \quad \text{at } (2,-1,1)\end{aligned}$$

and of magnitude= $|\nabla F| = \sqrt{4y^2 + 4x^2 + 4z^2} = 2\sqrt{6}$ at(2,-1,1).

Equations of the tangent line and normal plane to a space curve:

Consider a space curve $x=f(u)$,$y=g(u)$,$z=h(u)$.If

$$\underline{R} = f(u)\underline{i} + g(u)\underline{j} + h(u)\underline{k},$$

then a vector tangent to the curve at point p(where $u=u_0$)is given by

$$\underline{T}_0 = d\underline{R}/du\big|_p \qquad ...(3.11)$$

Then if $\underline{r}_0$ and $\underline{r}$ are the vectors drawn respectively from O to P and Q on the tangent line, we have

$$(\underline{r}-\underline{r}_0)\times T_0 = 0 \qquad (\underline{r}-\underline{r}_0) \wedge \underline{T}_0 = 0 \qquad \ldots(3.12)$$

since $(\underline{r}-\underline{r}_0)$ and $\underline{T}_0$ are collinear.

In rectangular form this becomes

$$\frac{(x-x_0)}{f'(u_0)} = \frac{(y-y_0)}{g'(u_0)} = \frac{(z-z_0)}{h'(u_0)} \qquad \ldots(3.13)$$

If $\underline{r}$ is the vector from O to any point (x,y,z) on the normal plane, it follows that $(\underline{r}-\underline{r}_0)$ is perpendicular to T_0, i.e.

$$(\underline{r}-\underline{r}_0)\cdot T_0 = 0 \qquad \ldots(3.14)$$

or in rectangular form

$$f'(u_0)(x-x_0)+g'(u_0)(y-y_0)+h'(u_0)(z-z_0)=0 \qquad \ldots(3.15)$$

Ex.3.7: Find the equations of the tangent line and normal plane to the space curve $x=6\sin t$, $y=4\cos 3t$, $z=2\sin 5t$, at the point where $t=\pi/4$.

Let $\underline{R}= 6\sin t\,\underline{i}+ 4\cos 3t\,\underline{j} + 2\sin 5t\,\underline{k}$ be the position vector of a point p on the curve. The equation of the tangent line to the curve at point where $t=\pi/4$ is

$$\frac{x-6/\sqrt{2}}{6/\sqrt{2}} = \frac{y+4/\sqrt{2}}{-12/\sqrt{2}} = \frac{z+2/\sqrt{2}}{-10/\sqrt{2}} \quad \text{at} \quad t=\pi/4$$

The equation of the normal plane, at the same point, is

$$\frac{6}{\sqrt{2}}(x-6/\sqrt{2}) - \frac{12}{\sqrt{2}}(y+4/\sqrt{2}) - \frac{10}{\sqrt{2}}(z+2/\sqrt{2}) = 0 \quad \text{at } t=\pi/4$$

or

$$3x-6y-5z=26\sqrt{2}.$$

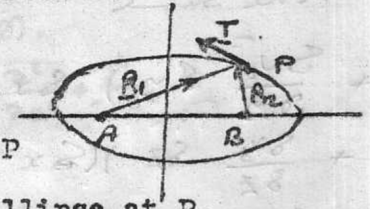
Ex.3.8: Let R be the distance from a fixed point A(a,b,c) to any point p(x,y,z). Show that ∇R is a unit vector in the direction $\underline{AP} = \underline{R}$.

If $\underline{r}_A$ and $\underline{r}_P$ are the position vectors $a\underline{i} + b\underline{j} + c\underline{k}$ and $x\underline{i} + y\underline{j} + z\underline{k}$ of A and P respectively, then $\underline{R}=(x-a)\underline{i}+(y-b)\underline{j}+(z-c)\underline{k}$, so that $R = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$. Then

$$\nabla R = \nabla\left(\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}\right) = \frac{(x-a)\underline{i}+(y-b)\underline{j}+(z-c)\underline{k}}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$$

$= \underline{R} / R$ is a unit vector in the direction R.

Ex.3.9: Let P be any point on an ellipse whose foci are at points A and B, prove that lines AP&BP make equal angles with the tangent to the ellipse at P.



Let $\underline{R}_1=\underline{AP}$ and $\underline{R}_2=\underline{BP}$ denote vectors drawn respectively from foci A and B to point P on the ellipse, and let $\underline{T}$ be a unit tangent to the ellipse at P.

Since an ellipse is the the locus of all points P the sum of whose distances from two fixed points A and B is a constant c, it is seen that the equation of the ellipse is $R_1+R_2=c$.

Since $\nabla(R_1+R_2)$ is a normal to the ellipse; hence

$$\nabla(R_1+R_2)\cdot\underline{T} = 0 \quad \text{or} \quad \nabla R_2 \cdot \underline{T} = -\nabla R_1\cdot\underline{T}$$

Since ∇R_1 and ∇R_2 are unit vectors in directions R_1 and R_2 respectively, the cosine of the angle between ∇R_2 and $\underline{T}$ is equal to the cosine of the angle between ∇R_1 and $-\underline{T}$; hence the angles themselves are equal. Reflection of waves is a physical interpretation of the example.

3.3 The Divergence of a Vector Field:

Let $\underline{V}(x,y,z) = V_1\underline{i}+V_2\underline{j}+V_3\underline{k}$ be defined and differentiable at each point(x,y,z) in a certain region of space(i.e.$\underline{V}$ defines a differentiable Vector Field).Then the divergence of V,written

$\nabla\cdot\underline{V}$ or div $\underline{V}$, is defined by

$$\nabla\cdot\underline{V} = \left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)\cdot\left(V_1\underline{i}+V_2\underline{j}+V_3\underline{k}\right)$$

$$= \frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z} \qquad \ldots(3.16)$$

Note the analogy with $\underline{A}.\underline{B} = A_1B_1+A_2B_2+A_3B_3$.Also note that

$$\nabla\cdot\underline{V} \neq \underline{V}\cdot\nabla$$

Ex.3.10 Given $\phi=2x^3y^2z^4$,(a)Find $\nabla.\nabla\phi$(or div grad ϕ).

(b)Show that $\nabla.\nabla\phi = \nabla^2\phi$,where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

$$(a)\ \nabla\phi = \underline{i}\frac{\partial}{\partial x}(2x^3y^2z^4)+\underline{j}\frac{\partial}{\partial y}(2x^3y^2z^4)+\underline{k}\ (2x^3y^2z^4)$$

$$= 6x^2y^2z^4\underline{i}+4x^3yz^4\underline{j}+8x^3y^2z^3\underline{k}$$

Then $$\nabla.\nabla\phi = \left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)\cdot\left(6x^2y^2z^4\underline{i}+4x^3yz^4\underline{j}+8x^3y^2z^3\underline{k}\right)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4)+\frac{\partial}{\partial y}(4x^3yz^4)+\frac{\partial}{\partial z}(8x^3y^2z^3)$$

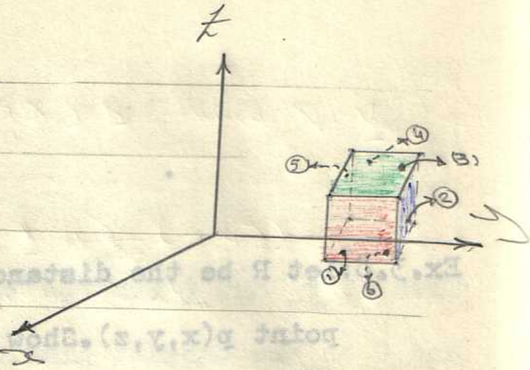
$$= 12xy^2z^4+4x^3z^4+24x^3y^2z^2$$

$$(b)\ \nabla.\nabla\phi = \left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\underline{i}+\frac{\partial\phi}{\partial y}\underline{j}+\frac{\partial\phi}{\partial z}\underline{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

-9- lecture



1) $(\delta y\, \delta z)\, \underline{i} = \delta \underline{S}_1$

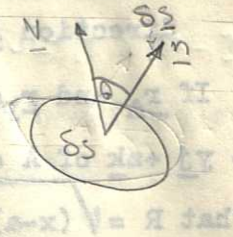
4) $(\delta y\, \delta z)(-\underline{i}) = \delta \underline{S}_4$

2) $(\delta x\, \delta z)\, \underline{j} = \delta \underline{S}_2$

5) $(\delta x\, \delta z)(-\underline{j}) = \delta \underline{S}_5$

3) $(\delta x\, \delta y)\, \underline{k} = \delta \underline{S}_3$

(6) $(\delta x\, \delta y)(-\underline{k}) = \delta \underline{S}_6$



Let $\underline{V} = V_1 \underline{i} + V_2 \underline{j} + V_3 \underline{k}$

Flux $V_n \delta S = \underline{V} \cdot \delta \underline{S} = N \cos\theta\, \delta S$

& I conf on (1)

$$\left(V_1 + \frac{\partial V_1}{\partial x}\delta x\right)\delta y\, \delta z + (V_1 + 0)(-\delta y\, \delta z) = \frac{\delta V_1}{\partial \delta}\delta x\, \delta y\, \delta z \quad (1)$$

& the same

$$\left(V_2 + \frac{\delta V_2}{\delta y}\delta y\right)\delta x\, \delta z + (V_2(-\delta x\, \delta z)) = \frac{\delta V_2}{\delta y}\delta x\, \delta y\, \delta z \quad (2)$$

$$\left(V_3 + \frac{\delta V_3}{\delta z}\delta z\right)(\delta x\, \delta y) + V_3(-\delta x\, \delta y) = \frac{\delta V_3}{\delta z}\delta x\, \delta y\, \delta z \quad (3)$$

eqn ① $= V_1 \underline{i} \cdot \delta S_x$

eqn ② $= V_2 \underline{j} \cdot \delta S_y$

eq. ③ $= V_3 \underline{k} \cdot \delta S_z$

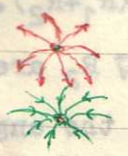
$$\underline{V} \cdot \delta \underline{S} = (V_1 \underline{i} + V_2 \underline{j} + V_3 \underline{k}) \cdot (\delta S_x \underline{i} + \delta S_y \underline{j} + \delta S_z \underline{k})$$

$$= (V_1 \underline{i} \cdot \delta S_x) + (\delta V_2 \cdot \delta S_y) + (\delta V_3 \cdot \delta S_z) =$$

Flux $= \left(\frac{\delta V_1}{\delta x} + \frac{\delta V_2}{\delta y} + \frac{\delta V_3}{\delta z}\right)\delta x\, \delta y\, \delta z$; as $\delta v \to 0$ 2D

Flux / volume $= \frac{\partial V_1}{\delta x} + \frac{\partial V_2}{\partial y} + \frac{\delta V_3}{\partial z} = \nabla \cdot \underline{V}$

$\text{div}\, \underline{V} = \nabla \cdot \underline{V}$
- > 0 divergence (+ve) Source
- < 0 divergence (−ve) Sink
- $= 0$ Solenoidal $\rightrightarrows$ (as in a tube)



Ex.3.11: Prove that $\nabla^2(1/r) = 0$

$$\nabla^2(1/r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\& \quad \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0$$

The equation $\nabla^2\phi = 0$ is called Laplace's equation. It follows that $\phi = 1/r$ is a solution of this equation.

Ex.3.12: Prove $\nabla\cdot(\phi\underline{A}) = (\nabla\phi)\cdot\underline{A} + \phi(\nabla\cdot\underline{A})$.

$$\nabla\cdot(\varphi\underline{A}) = \nabla\cdot(\varphi A_1\underline{i} + \varphi A_2\underline{j} + \varphi A_3\underline{k})$$

$$= \frac{\partial}{\partial x}(\varphi A_1) + \frac{\partial}{\partial y}(\varphi A_2) + \frac{\partial}{\partial z}(\varphi A_3)$$

$$= \frac{\partial\varphi}{\partial x}A_1 + \varphi\frac{\partial A_1}{\partial x} + \frac{\partial\varphi}{\partial y}A_2 + \varphi\frac{\partial A_2}{\partial y} + \frac{\partial\varphi}{\partial z}A_3 + \varphi\frac{\partial A_3}{\partial z}$$

$$= \left(\frac{\partial\varphi}{\partial x}A_1 + \frac{\partial\varphi}{\partial y}A_2 + \frac{\partial\varphi}{\partial z}A_3\right) + \varphi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$\nabla\cdot(\varphi\underline{A}) = \nabla\varphi\cdot\underline{A} + \varphi\,\nabla\cdot\underline{A}$$

Ex: If $\underline{V} = 3x\,\underline{i} + (y^2 - Z)\,\underline{j} + 5xy\,\underline{k}$ find $\nabla \cdot \underline{V}$

Sol:

$$\nabla \cdot \underline{V} = \frac{\partial}{\partial x} 3x + \frac{\partial}{\partial y}(y^2 - Z) + \frac{\partial}{\partial z}(5xy)$$

$= 3 + 2y$ and for the given point

$\therefore \nabla \cdot \underline{V} = 7$

Note

$$\underline{V} \cdot \nabla = (v_1\,\underline{i} + v_2\,\underline{j} + v_3\,\underline{k}) \cdot \left(\frac{\partial}{\partial x}\,\underline{i} + \frac{\partial}{\partial y}\,\underline{j} + \frac{\partial}{\partial z}\,\underline{k}\right)$$

$$= \left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y} + v_3 \frac{\partial}{\partial z}\right) =$$

$$3x \frac{\partial}{\partial x} + (y^2 - Z)\frac{\partial}{\partial y} + 5xy \frac{\partial}{\partial z}\Big|_{1,2,3}$$

$$= 3\frac{\partial}{\partial x} + (1)\frac{\partial}{\partial y} + 10\frac{\partial}{\partial z}.$$

Note: Although $A.B = B.A$

but $\nabla \cdot \underline{V} \not\equiv \underline{V} \cdot \nabla$

Let $\Phi(x, y, Z)$, $\nabla \Phi = \underline{V} = v_1\,\underline{i} + v_2\,\underline{j} + v_3\,\underline{k} = \frac{\partial \Phi}{\partial x}\,\underline{i} + \frac{\partial \Phi}{\partial y}\,\underline{j} + \frac{\partial \Phi}{\partial z}\,\underline{k}$

$$\nabla \cdot \underline{V} = \nabla \cdot \nabla \Phi = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right)$$

$$\therefore \nabla \cdot \nabla \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial Z^2} = \nabla^2 \cdot \Phi$$

"Laplacian"

where ∇^2 is the 2nd differential operator)

$$\therefore \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial Z^2} \;;\; \nabla \equiv \frac{\partial}{\partial x}\,\underline{i} + \frac{\partial}{\partial y}\,\underline{j} + \frac{\partial}{\partial z}\,\underline{k}$$

Ex.3.13: Prove $\nabla\cdot(\underline{r}/r^3) = 0$

Let $\phi = r^{-3}$ and $\underline{A} = \underline{r}$ in the result of the previous example, then

$$\nabla\cdot(r^{-3}\underline{r}) = \nabla r^{-3}\cdot\underline{r} + r^{-3}\nabla\cdot\underline{r}$$

$$= -3r^{-5}\,\underline{r}\cdot\underline{r} + 3r^{-3}$$

$$= 0$$

A vector whose divergence is zero is sometimes called solenoidal.

Ex.3.14: Show that $\underline{V} = (2x^2+8xy^2z)\underline{i} + (3x^3y-3xy)\underline{j} - (4y^2z^2+2x^3z)\underline{k}$ is not solenoidal. Use the relation given in Ex.3.12 to show that $xyz^2\underline{V}$ is solenoidal.

$$\nabla\cdot\underline{V} = \frac{\partial}{\partial x}(2x^2+8xy^2z) + \frac{\partial}{\partial y}(3x^3y-3xy) - \frac{\partial}{\partial z}(4y^2z^2+2x^3z)$$

$$= x^3 + x \neq 0$$

let $\varphi = xyz^2$, Then $\nabla\cdot(\varphi\underline{V}) = \nabla\varphi\cdot\underline{V} + \varphi(\nabla\cdot\underline{V})$

i.e. $$\nabla\cdot(xyz^2\underline{V}) = (yz^2\underline{i} + xz^2\underline{j} + 2xyz\underline{k})\cdot\big[(2x^2+8xy^2z)\underline{i} + (3x^3y-3xy)\underline{j} - (4y^2z^2+2x^3z)\underline{k}\big]$$

$$+ (xyz^2)(x^3+x)$$

$$= 2x^2yz^2 + 8xy^2z^3 + 3x^4yz^2 - 3x^2yz^2 - 8xy^3z^2 - 4x^4yz^2 + x^4yz^2 + x^2yz^2 = 0$$

Ex: $2x^3 y^2 z^4$; find $\nabla\phi$ & $\nabla^2\phi$ at (1,2,-1)

Soln

$$\nabla\phi = 6x^2y^2z^4\,\underline{i} + 4x^3yz^4\,\underline{j} + 8x^3y^2z^3\,\underline{k}$$

$$\nabla\phi\Big|_{(1,2,-1)} = 24\,\underline{i} + 8\,\underline{j} - 32\,\underline{k}$$ R.T.? ©

$$\nabla^2\phi = 12xy^2z^4 + 4x^3z^4 + 24x^2y^2z^2$$

$$\nabla^2\phi\Big|_{(1,2,-1)} = 48 + 4 + 96 = 148$$

Ex: find $\nabla^2\left(\frac{1}{r}\right)$

Soln $$\nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\nabla\left(\frac{1}{r}\right) = \nabla\cdot\left[-r^{-3}\underline{r}\right]$$

$$= -\nabla\cdot\left(\frac{\underline{r}}{r^3}\right) = -\nabla\cdot\left\{\frac{x\underline{i} + y\underline{j} + z\underline{k}}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$= -\left[\frac{3r^3 - 3r^2\cdot r}{(x^2+y^2+z^2)}\right] = 0$$

$$\nabla^2\phi = 0$$ {LaPlace EQUATION}

Theory:

Let ϕ Scalar ; $\underline{V}$ vector $= V_1\underline{i} + V_2\underline{j} + V_3\underline{k}$

$$\phi\underline{V} = \phi V_1\,\underline{i} + \phi V_2\,\underline{j} + \phi V_3\,\underline{k}$$

$$\nabla\cdot(\phi\underline{V}) = \frac{\partial}{\partial x}(\phi V_1) + \frac{\partial}{\partial y}(\phi V_2) + \frac{\partial}{\partial z}(\phi V_3)$$

$$\nabla\cdot(\phi\underline{V}) = V_1\frac{\partial\phi}{\partial x} + V_2\frac{\partial\phi}{\partial y} + V_3\frac{\partial\phi}{\partial z} + \phi\frac{\partial V_1}{\partial x} + \phi\frac{\partial V_2}{\partial y} + \phi\frac{\partial V_3}{\partial z}$$

$$= (V_1\underline{i} + V_2\underline{j} + V_3\underline{k})\cdot\frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k} + \phi\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) \Rightarrow$$

$$\left\{\nabla\cdot(\phi\underline{V}) = \underline{V}\cdot\nabla\phi + \phi\,\nabla\cdot\underline{V}\right\}$$

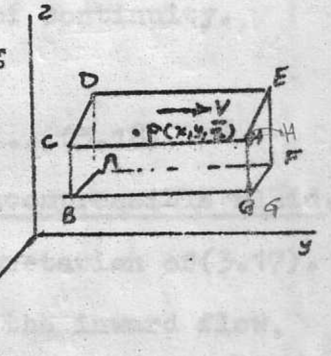
V.Imp

(R)

The Equation of Continuity:

(Consider a fluid of density $\rho(x,y,z,t)$ and velocity $\underline{V}(x,y,z,t)$.) If S is the representative vector of the area of the plane surface, then $\rho\underline{V}.\underline{S}$ is the mass of fluid flowing through the surface S in a unit time.

Take a small parallelepiped, of the fluid, having centre at $p(x,y,z)$ and edges parallel to the coordinate axes and having magnitude Δx, Δy, Δz respectively. Then we have



x component of velocity v at p = V_1

x component of V at centre of AFED $=V_1-\frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x$ approx.

x component of V at centre of GHCB $=V_1+\frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x$ approx.

Then the volume of fluid crossing AFED per unit time $=(V_1-\frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x)\Delta y\Delta z$

volume of fluid crossing GHCB per unit time $=(V_1+\frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x)\Delta y\Delta z$

Thus the gain in volume per unit time in x direction $=\frac{\partial V_1}{\partial x}\Delta x\Delta y\Delta z$

Similarly, gain in volume per unit time in y direction

$$=\frac{\partial V_2}{\partial y}\Delta x\Delta y\Delta z$$

gain in volume per unit time in z direction

$$=\frac{\partial V_3}{\partial z}\Delta x\Delta y\Delta z$$

The total gain in volume per unit volume per unit time

$$=\frac{(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z})\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z} = \frac{\text{mass}}{\text{Volume}}$$

$$= \operatorname{div}\underline{V} = \nabla.\underline{V}$$

Problem Show that the vector

$$\underline{V} = (2x^2 + 8xy^2z)\underline{i} + (3x^3y - 3xy)\underline{j} - (4y^2z^2 + 2x^3z)\underline{k}$$

is not Solenoidal i.e $\nabla \cdot V \neq 0$

and $xyz^2\underline{V}$ is Solenoidal $(\nabla \cdot (S\underline{V}) = 0)$

prove $\nabla \cdot (S\underline{V}) = 0 = S(\nabla \cdot \underline{V}) + \underline{V} \cdot \nabla S$

By the principle of conservation of matter, the total gain in volume per unit volume per unit time must be equal to the time rate of decrease of density, i.e.

volume

$$\boxed{\text{density inside } \Delta D}\ \boxed{\nabla\cdot\underline{V}} = \boxed{-\frac{\partial\rho}{\partial t}} \rightarrow \boxed{\text{density decrease outside } \Delta D} \quad \ldots(3.17)$$

This is known, in hydrodynamics, as the equation of continuity.

If there is no gain of fluid anywhere, then

$$\nabla\cdot\underline{V} = 0 \qquad \ldots(3.18)$$

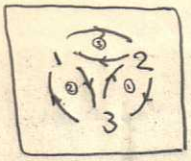
This is called the continuity equation for an incompressible fluid.

The name divergence originated inthe interpretation of(3.17). $\nabla\cdot\underline{V}$ represents the excess of the outward over the inward flow, or the divergence of the fluid.

If the divergence exists at a point in a fluid, whether liquid or gas, and is positive, it expresses the rate at which fluid is flowing away from the point per unit volume. Hence, either the fluid is expanding and its density is falling with time, or the point is a source, at which fluid is entering the field.

If the divergence is negative(convergerce), the fluid is contracting and its density is rising at the point, or the point is a negative source (a sink), at which fluid is leaving the field.

If the divergence is zero, the flux entering any element of space is exactly balanced by that leaving it. Since fluid is neither created nor destroyed at any point, it is said to have no sources or sinks. The fluid is incompressible(Eq.3.18). A vector V whose divergence is zero is a solenoidal field vector(no source or sink).



(7) 3.4 The Curl of a Vector Field:

If $\underline{V}(x,y,z)$ is a differentiable vector field then the curl or rotation of $\underline{V}$, written $\nabla\wedge\underline{V}$, Curl $\underline{V}$ or rot $\underline{V}$, is defined by

$$\nabla\wedge\underline{V}=\left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)\wedge\left(V_1\underline{i}+V_2\underline{j}+V_3\underline{k}\right)$$

$$\frac{\text{Circulation}}{\text{area}}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_1 & V_2 & V_3\end{vmatrix}\qquad\ldots(3.19)$$

Consider a vector field $\underline{V}$, which cannot be expressed as a gradient of a scalar field, i.e. in which the line integral round any closed path is not zero. We have a small vector area of any shape at any point in the vector field and we compute the line integral of the vector $\underline{V}$ around its bounding edge, there will be an orientation of the area for which the line integral is greatest. The amount of this maximum line integral per unit area is called the curl of the vector field at the point.

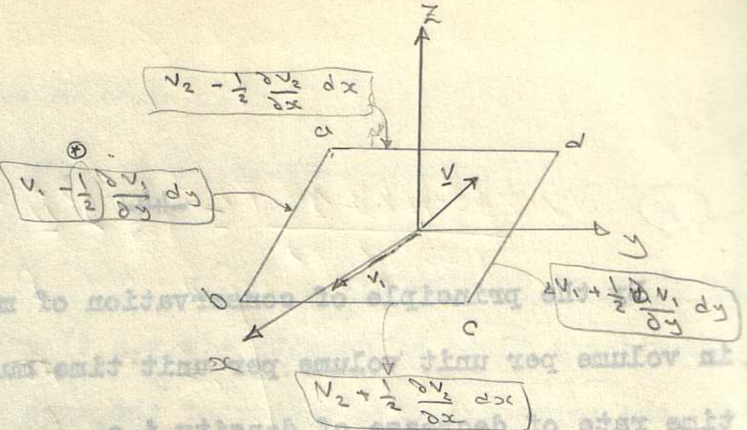
However, a field $\underline{V}$ which has a potential ϕ say (i.e. $\underline{V}=\nabla\phi$),

$\overset{5}{\text{V.Imp,}}$ (Curl $\underline{V}=\nabla\wedge\nabla\phi=0$) and the field is irrotational. At each point, the field is normal to the curve $\phi=$const. which passes through this point. The field, in this case, is referred to as laminar, since it divides the regions into layers or laminae.

the vector, which is the gradiant of scalar quantity, is an irrotational; and the scalar quantity is the potential of this field. $\nabla\times\nabla\phi=0$ — $\underset{5}{\text{V.Imp,}}$

lecture

(*) the one half $\frac{1}{2}$ denotes the half the original change occured


Z
y
x
a
b
c
d
$\underline{V}$
V_1
$V_2 - \frac{1}{2}\frac{\partial V_2}{\partial x}dx$
(*)
$V_1 - \frac{1}{2}\frac{\partial V_1}{\partial y}dy$
$V_1 + \frac{1}{2}\frac{\partial V_1}{\partial y}dy$
$V_2 + \frac{1}{2}\frac{\partial V_2}{\partial x}dx$

Elements of Circulation

$\underline{V}\cdot d\underline{r} = V\cos\theta\, dr$

ab : $(V_1 - \frac{1}{2}\frac{\partial V_1}{\partial y}dy)\,dx$

bc : $(V_2 + \frac{1}{2}\frac{\partial V_2}{\partial x}dx)\,dy$

cd : $(V_1 + \frac{1}{2}\frac{\partial V_1}{\partial y}dy)(-dx)$ {(-ve) for it is in the -ve direction for the two axes x, y}

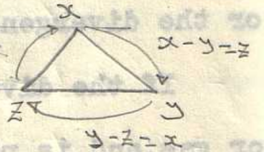
da : $(V_2 - \frac{1}{2}\frac{\partial V_2}{\partial x}dx)(-dy)$

$$\text{Circulation} = \left(V_1 - \frac{1}{2}\frac{\partial V_1}{\partial y} - V_1 - \frac{1}{2}\frac{\partial V_1}{\partial y}\right)dx + \left(V_2 + \frac{1}{2}\frac{\partial V_2}{\partial x} - V_2 + \frac{1}{2}\frac{\partial V_2}{\partial x}\right)dy = \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)dx\,dy$$

$$\left.\frac{\text{Circulation}}{\text{area}}\right|_z = \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

z (i.e ⊥ z plane)

(Note) etc



$$\left.\frac{\text{Circulation}}{\text{area}}\right|_x = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)$$

$$\left.\frac{\text{Circulation}}{\text{area}}\right|_y = \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)$$

$$\text{from (3.19)} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\underline{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\underline{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\underline{k} = \nabla \cdot \underline{V} = $$

when $\nabla \cdot \underline{V} = 0$ then the vector is irrotational

$\nabla \times \nabla\phi$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = (0-0)\underline{i} + (0-0)\underline{j} + (0-0)\underline{k}$$

i.e if the rotation of a vector = 0 then this vector must be $\nabla\phi$

Ex.3.15: Prove $\nabla\wedge(\phi\underline{A}) = (\nabla\phi)\wedge\underline{A} + \phi(\nabla\wedge\underline{A})$

$$\nabla\wedge(\phi\underline{A}) = \nabla\wedge(\phi A_1\underline{i} + \phi A_2\underline{j} + \phi A_3\underline{k})$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\underline{i} + \left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right]\underline{j}$$

$$+ \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]\underline{k}$$

$$= \phi\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi(\nabla\wedge\underline{A}) + (\nabla\phi)\wedge\underline{A}$$

Ex.3.16: Prove $\nabla\wedge(\nabla\phi) = 0$ (curl grad $\phi = 0$)

$$\nabla\wedge(\nabla\phi) = \nabla\wedge\left(\frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}\right)$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\underline{i} + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)\underline{j} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\underline{k}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

From the Central forces i.e.

$$\underline{F} = \frac{\underline{r}}{r^3} = \frac{\underline{r}}{r} \times \frac{1}{r^2} = \underline{u}\left(\frac{1}{r^2}\right) = \frac{x\underline{i} + y\underline{j} + z\underline{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ \frac{x}{\text{denominator}} & \frac{y}{\text{denomi.}} & \frac{z}{\text{denomi.}} \end{vmatrix}$$

$$\nabla \times \underline{F} = \left[\frac{-z \times \frac{3}{2}\cancel{(x^2+y^2+z^2)^{1/2}} \times 2y}{\cancel{(x^2+y^2+z^2)^{3/2}}} - \frac{-y \times \frac{3}{2}\cancel{(x^2+y^2+z^2)^{1/2}} \times 2z}{\text{denominator}} \right] \underline{i} +$$

$$\left[\overset{0}{\diagup} \right] \underline{j} + \left[\overset{0}{\diagup} \right] \underline{k}$$

$$\therefore \nabla \times \underline{F} = 0 \quad \text{if} \quad \underline{F} = \frac{\underline{r}}{r^3}$$

i.e every force which obeys the inverse square law is an irrotational force i.e. its force field is irrotational.

$$\nabla \times \underline{F} = 0 \Rightarrow \underline{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \phi = \int \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx + C_1(y,z)$$

$$= -\frac{1}{(x^2+y^2+z^2)^{1/2}} + C_1(y,z) \; ; \; \therefore \phi = -\frac{1}{r}$$

let $\underline{F} = S\underline{V}$; $\underline{V} = V_1\underline{i} + V_2\underline{j} + V_3\underline{k}$

$$\nabla \times \underline{F} = \nabla \times (S\underline{V}) =$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ SV_1 & SV_2 & SV_3 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} & \frac{\partial S}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$\therefore \left(\nabla \times (S\underline{V}) = (\nabla S \times \underline{V}) + S\,\nabla \times \underline{V} \right) \quad \text{(V.I mp)}$$

$$\therefore \nabla \times \underline{F} = \nabla \wedge \left(\frac{\underline{r}}{r^3}\right) \times \underline{r} + \frac{1}{r^3}(\nabla \wedge \underline{r}) = \frac{-3}{r^5}(\underline{r} \times \underline{r}) + \frac{1}{r^3}(0) = 0$$

Ex.3.17: A vector is called irrotational if curl $\underline{V}=0$. Find constants a,b,c so that

$$\underline{V} = (x+2y+az)\underline{i} + (bx-3y-z)\underline{j} + (4x+cy+2z)\underline{k}$$

is irrotational. Show that $\underline{V}$ can be expressed as the gradient of a scalar function.

$$\nabla\wedge\underline{V} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$$

This equals zero when $a=4, b=2, c=-1$ and

$$\underline{V} = (x+2y+4z)\underline{i} + (2x-3y-z)\underline{j} + (4x-y+2z)\underline{k}$$

Assume $\underline{V} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\underline{i} + \frac{\partial\varphi}{\partial y}\underline{j} + \frac{\partial\varphi}{\partial z}\underline{k}$

Then (i) $\frac{\partial\varphi}{\partial x} = x+2y+4z$, (ii) $\frac{\partial\varphi}{\partial y} = 2x-3y-z$, (iii) $\frac{\partial\varphi}{\partial z} = 4x-y+2z$

Integrating (i) partially with respect to x, keeping y and z const.,

$$\varphi = \frac{x^2}{2} + 2xy + 4xz + f(y,z) \qquad \text{(iv)}$$

where f(y,z) is an arbitrary function of y and z. Similarly from (ii) and (iii)

$$\varphi = 2xy - \frac{3y^2}{2} - yz + g(x,z) \qquad \text{(v)}$$

$$\varphi = 4xz - yz + z^2 + h(x,y) \qquad \text{(vi)}$$

Comparison of (iv), (v) and (vi) shows that there will be a common value of ∅ if we choose

$$f(y,z) = -\frac{3y^2}{2} + z^2, \quad g(x,z) = \frac{x^2}{2} + z^2, \quad h(x,y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that $\varphi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$

In general if $\nabla\wedge\underline{V}=0$, then we can find ∅ so that $\underline{V}=\nabla\emptyset$. A vector field V which can be derived from a scalar fiold ∅ so that $\underline{V}=\nabla\emptyset$ is called a conservative vector field and ∅ is called the scalar potential. Note that conversely if $\underline{V}=\nabla\emptyset$, then $\nabla\wedge\underline{V}=0$.

Note Given a central force $\underline{F}$, thus it can be proved that its field is irrotational i.e it is possible to prove in two ways that $\nabla \wedge \underline{F} = 0$ (or) $\nabla \wedge (S\underline{V}) = (\nabla S \wedge \underline{V}) + S \nabla \wedge \underline{V}$ and in this case the field is called conservative which means that it represents the force $\underline{F}$ as the gradient of a scalar quantity ϕ, this quantity is the potential of this field. The value of this potential can be deduced by integrating the components of $\nabla \phi$

$$d\phi = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}$$

; make sure that ϕ resulted is the potential of this field by finding $\nabla \phi$ and equalizing it by $\underline{F}$.

Ex.3.18: Prove $\nabla.(\nabla\wedge\underline{A}) = 0$ (div curl $\underline{A} = 0$).

$$\nabla.(\nabla\wedge\underline{A}) = \nabla.\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\partial y} - \frac{\partial^2 A_2}{\partial x\partial z} + \frac{\partial^2 A_1}{\partial y\partial z} - \frac{\partial^2 A_3}{\partial y\partial x} + \frac{\partial^2 A_2}{\partial z\partial x} - \frac{\partial^2 A_1}{\partial z\partial y} = 0$$

assuming that A has continuous second partial derivatives.

Ex.3.19: Prove $\nabla\wedge(\nabla\wedge\underline{A}) = -\nabla^2\underline{A} + \nabla(\nabla.\underline{A})$

$$\nabla\wedge(\nabla\wedge\underline{A}) = \nabla\wedge\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \end{vmatrix}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1\underline{i} + A_2\underline{j} + A_3\underline{k})$$

$$+ \underline{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \underline{j}\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$+ \underline{k}\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2\underline{A} + \nabla\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) = -\nabla^2\underline{A} + \nabla(\nabla.\underline{A})$$

Ex.3.20: If $\underline{V} = \underline{w} \wedge \underline{r}$, prove $\underline{w} = \frac{1}{2}$ curl $\underline{V}$ where $\underline{w}$ is a constant vector.

$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$

$$\text{Curl}\,\underline{V} = \nabla\wedge\underline{V} = \nabla\wedge(\underline{\omega}\wedge\underline{r}) = \nabla\wedge\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= 2(\omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}) = 2\underline{\omega}$$

$\underline{E}$ ≡ electric field

$\underline{H}$ ≡ magnetic "

Ex.3.21: If $\nabla\cdot\underline{E} = 0$, $\nabla\cdot\underline{H} = 0$, $\nabla\wedge\underline{E} = -\frac{\partial H}{\partial t}$, $\nabla\wedge\underline{H} = \frac{\partial E}{\partial t}$

to deduce the wave equation

show that $\underline{E}$ and $\underline{H}$ satisfy $\overset{(1)}{\nabla^2} u = \overset{(2)}{\partial^2} u / \partial t^2$

(1) 2nd differential w.r. to space

(2) 2nd differential w.r. to time

$$\nabla\wedge(\nabla\wedge\underline{E}) = \nabla\wedge\left(-\frac{\partial H}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\wedge\underline{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial\underline{E}}{\partial t}\right) = -\frac{\partial^2 E}{\partial t^2}$$

But $\nabla\wedge(\nabla\wedge\underline{E}) = -\nabla^2\underline{E} + \nabla(\nabla\cdot\underline{E}) = -\nabla^2\underline{E}$, Then $\nabla^2\underline{E} = \frac{\partial^2 E}{\partial t^2}$

Also $\nabla\wedge(\nabla\wedge\underline{H}) = \nabla\wedge\left(\frac{\partial E}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\wedge\underline{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial H}{\partial t}\right) = -\frac{\partial^2 H}{\partial t^2}$

But $\nabla\wedge(\nabla\wedge\overset{H}{\underline{H}}) = -\nabla^2\underline{H} + \nabla(\nabla\cdot H) = -\nabla^2\underline{H}$, Then $\nabla^2\underline{H} = \frac{\partial^2 H}{\partial t^2}$

The given equations are related to Maxwell's equations of electromagnetic theory. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$ is called the wave equation.

3.5 Formulae Involving the Operator :

i- $\nabla(\varphi+\psi)=\nabla\varphi+\nabla\psi$ grad$(\varphi+\psi)$=grad φ +grad ψ

ii- $\nabla\cdot(\underline{A}+\underline{B})=\nabla\cdot\underline{A}+\nabla\cdot\underline{B}$ div$(\underline{A}+\underline{B})$ =div $\underline{A}$ +div $\underline{B}$

iii- $\nabla\wedge(\underline{A}+\underline{B})=\nabla\wedge\underline{A}+\nabla\wedge\underline{B}$ curl$(\underline{A}+\underline{B})$=curl $\underline{A}$ +curl $\underline{B}$

iv- $\nabla\cdot(\varphi\underline{A})=(\nabla\varphi)\cdot\underline{A}+\varphi(\nabla\cdot\underline{A})$

v- $\nabla\wedge(\varphi\underline{A})=\nabla\varphi\wedge\underline{A}+\varphi(\nabla\wedge\underline{A})$

vi- $\nabla\cdot(\underline{A}\wedge\underline{B})=\underline{B}\cdot(\nabla\wedge\underline{A})-\underline{A}\cdot(\nabla\wedge\underline{B})$

vii- $\nabla\wedge(\underline{A}\wedge\underline{B})=(\underline{B}\cdot\nabla)\underline{A}-\underline{B}(\nabla\cdot\underline{A})-(\underline{A}\cdot\nabla)\underline{B}+\underline{A}(\nabla\cdot\underline{B})$

viii- $\nabla(\underline{A}\cdot\underline{B})=(\underline{B}\cdot\nabla)\underline{A}+(\underline{A}\cdot\nabla)\underline{B}+\underline{B}\wedge(\nabla\wedge\underline{A})+\underline{A}\wedge(\nabla\wedge\underline{B})$

ix- $\nabla\cdot\nabla\varphi=\nabla^2\varphi$ also $\nabla(\nabla\cdot\underline{A})=\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)\underline{i}+\text{-----}$

↗ the formula of how to deal with the grad

x- $\nabla\wedge(\nabla\varphi)=0$ The curl of the gradient of ϕ is zero.

xi- $\nabla\cdot(\nabla\wedge\underline{A})=0$ The divergence of the curl of A is zero.

xii- $\nabla\wedge(\nabla\wedge\underline{A})=\nabla(\nabla\cdot\underline{A})-\nabla^2\underline{A}$; (xiii) $\nabla\times(\phi\underline{A})=\phi\,\nabla\times\underline{A}+\nabla\phi\times A$

In Formulae ix-xii, it is supposed that ϕ and $\underline{A}$ have continuous second partial derivatives.

(xiv) $\nabla\times(\underline{A}\times\underline{B})=(\underline{B}\cdot\nabla)\underline{A}-(\underline{A}\cdot\nabla)\underline{B}+(\nabla\cdot\underline{B})\underline{A}-(\nabla\cdot\underline{A})\underline{B}$

Ex.3.22: Evaluate $\nabla\cdot(\underline{A}\wedge\underline{r})$ if $\nabla\wedge\underline{A}=0$.

Let $\underline{A}=A_1\underline{i}+A_2\underline{j}+A_3\underline{k}$ & $\underline{r}=x\underline{i}+y\underline{j}+z\underline{k}$

Then

$$\underline{A}\wedge\underline{r}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ A_1 & A_2 & A_3\\ x & y & z\end{vmatrix}$$

$$\nabla\cdot(\underline{A}\wedge\underline{r})=z\frac{\partial A_2}{\partial x}-y\frac{\partial A_3}{\partial x}+x\frac{\partial A_3}{\partial y}-z\frac{\partial A_1}{\partial y}+y\frac{\partial A_1}{\partial z}-x\frac{\partial A_2}{\partial z}$$

$$=(x\underline{i}+y\underline{j}+z\underline{k})\cdot\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\underline{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\underline{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\underline{k}\right]$$

$$=\underline{r}\cdot(\nabla\wedge\underline{A})=0 \quad \text{if} \quad \nabla\wedge\underline{A}=0$$

(i) $(\Phi+\Psi)'_x\,\underline{i} + (\Phi+\Psi)'_y\,\underline{j} + (\Phi+\Psi)'_z\,\underline{k} = \nabla\Phi + \nabla\Psi$ ①

(ii) $(\underline{A}+\underline{B})'_x + (\underline{A}+\underline{B})'_y + (\underline{A}+\underline{B})'_z = \nabla\cdot\underline{A} + \nabla\cdot\underline{B}$ ②

(iii) $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_1 & B_2 & B_3 \end{vmatrix} = \nabla\times A + \nabla\times B$ ③

(iv) $\dfrac{\partial\phi A_1}{\partial x} + \dfrac{\partial\phi A_2}{\partial y} + \dfrac{\partial\phi A_3}{\partial z} = A_1\dfrac{\partial\phi}{\partial x} + \underline{\phi\dfrac{\partial A_1}{\partial x}} + A_2\dfrac{\partial\phi}{\partial y} + \underline{\phi\dfrac{\partial A_2}{\partial y}} + A_3\dfrac{\partial\phi}{\partial z} + \underline{\phi\dfrac{\partial A_3}{\partial z}}$

$= \phi(\nabla\cdot\underline{A}) + \underline{A}\cdot(\nabla\phi)$

(v) $\nabla\times(\underline{A}\phi) = \begin{vmatrix} \underline{i} & \underline{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix} = \Big(\dfrac{\partial}{\partial y}(\phi A_3) - \dfrac{\partial}{\partial z}(\phi A_2),\ \dfrac{\partial}{\partial z}(\phi A_1) - \dfrac{\partial}{\partial x}\phi A_3,\ \dfrac{\partial}{\partial x}(\phi A_2) - \dfrac{\partial}{\partial y}(\phi A_1)\Big) =$

$\Big(\phi\dfrac{\partial A_3}{\partial y} + A_3\dfrac{\partial\phi}{\partial y}\Big) - \Big(\phi\dfrac{\partial A_2}{\partial z} + A_2\dfrac{\partial\phi}{\partial z}\Big),$

$\Big(\phi\dfrac{\partial A_1}{\partial z} + A_1\dfrac{\partial\phi}{\partial z}\Big) - \Big(\phi\dfrac{\partial A_3}{\partial x} + A_3\dfrac{\partial\phi}{\partial x}\Big),\ \Big(\phi\dfrac{\partial A_2}{\partial x} + A_2\dfrac{\partial\phi}{\partial x}\Big) - \Big(\phi\dfrac{\partial A_1}{\partial y} + A_1\dfrac{\partial\phi}{\partial y}\Big)$

$= \phi(\nabla\times\underline{A}) +$ ~~[illegible]~~ $(\nabla\phi)\times\underline{A} = \phi(\nabla\times\underline{A}) + (\nabla\phi)\times\underline{A}$

(vi) $\nabla\cdot(\underline{A}\times\underline{B}) =$ ~~[illegible]~~ $\nabla\cdot(A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1) =$

$\Big(A_2\dfrac{\partial B_3}{\partial x} + B_3\dfrac{\partial A_2}{\partial x}\Big) - \Big(A_3\dfrac{\partial B_2}{\partial x} + B_2\dfrac{\partial A_3}{\partial x}\Big) + \Big(A_3\dfrac{\partial B_1}{\partial y} + B_1\dfrac{\partial A_3}{\partial y}\Big) - \Big(A_1\dfrac{\partial B_3}{\partial y} + B_3\dfrac{\partial A_1}{\partial y}\Big) +$

$\Big(A_1\dfrac{\partial B_2}{\partial z} + B_2\dfrac{\partial A_1}{\partial z}\Big) - \Big(A_2\dfrac{\partial B_1}{\partial z} + B_1\dfrac{\partial A_2}{\partial z}\Big) =$

[1] $\Big[A_1\Big(\dfrac{\partial B_2}{\partial z} - \dfrac{\partial B_3}{\partial y}\Big) + A_2\Big(\dfrac{\partial B_3}{\partial x} - \dfrac{\partial B_1}{\partial z}\Big) + A_3\Big(\dfrac{\partial B_1}{\partial y} - \dfrac{\partial B_2}{\partial x}\Big)\Big] +$

[2] $\Big[B_1\Big(\dfrac{\partial A_3}{\partial y} - \dfrac{\partial A_2}{\partial z}\Big) + B_2\Big(\dfrac{\partial A_1}{\partial z} - \dfrac{\partial A_3}{\partial x}\Big) + B_3\Big(\dfrac{\partial A_2}{\partial x} - \dfrac{\partial B_1}{\partial y}\Big)\Big] =$

$-[-[1]] + [2] = \underbrace{\underline{B}\cdot(\nabla\times\underline{A})}_{[2]} - \underbrace{\underline{A}\cdot(\nabla\times\underline{B})}_{[-[1]]}$

(vii) $\nabla\times(\underline{A}\times\underline{B}) \Rightarrow \underline{A}\times\underline{B} = (A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1) = (C_1, C_2, C_3) \Rightarrow$

$\nabla\times\underline{C} = \dfrac{\partial}{\partial y}(A_1B_2 - A_2B_1) - \dfrac{\partial}{\partial z}(A_3B_1 - A_1B_3),\ \dfrac{\partial}{\partial z}(A_2B_3 - A_3B_2) - \dfrac{\partial}{\partial x}(A_1B_2 - A_2B_1),$

$\dfrac{\partial}{\partial x}(A_3B_1 - A_1B_3) - \dfrac{\partial}{\partial y}(A_2B_3 - A_3B_2) =$

Problems III-Gradient, Divergence & Curl

3.1: If $\phi(x,y,z) = 3x^2y-y^3z^2$, find $\nabla\phi$ at the point $(1,-2,-1)$.

Ans. $-12\underline{i}-9\underline{j}-16\underline{k}$

3.2: Prove $\nabla(F+G)=\nabla F+\nabla G$, and $\nabla(FG)=F\nabla G+G\nabla F$ where Fand G are differentiable scalar functions of x, y and z.

3.3: Find a unit normal to the surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

Ans. $(-\underline{i}+2\underline{j}+2\underline{k})/3$

3.4: Find an equation for the tangent plane to the surface $2xz^2-3xy-4x=7$ at the point $(1,-1,2)$.

Ans. $7x-3y+8z=26$

3.5: Find the directional derivative of $\phi=x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction $2i-j-2k$.

Ans. $37/3$

3.6: In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi=x^2yz^3$ a maximum? What is the magnitude of this maximum?

Ans. $-4\underline{i}-4\underline{j}+12\underline{k}$, $4\sqrt{11}$

3.7: Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Ans. $\cos\theta=\frac{8\sqrt{21}}{63}$

3.8: If $\underline{A}=x^2z\underline{i} - 2y^3z^2\underline{j} + xy^2z\underline{k}$, find $\nabla\cdot\underline{A}$ at the point (1, −1, 1).

Ans. −3

3.9: Given $\phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, find $\nabla^2\phi$

Ans. $6z+24xy-2z^3-6y^2z$

3.10: Prove $\nabla\cdot(\underline{A}+\underline{B}) = \nabla\cdot\underline{A} + \nabla\cdot\underline{B}$

3.11: Determine the constant a so that the vector $\underline{V}=(x+3y)\underline{i} + (y-2z)\underline{j} + (x+az)\underline{k}$ is solenoidal.

Ans. −2

3.12: If $\underline{A}=xz^3\underline{i} - 2x^2yz\underline{j} + 2yz^4\underline{k}$, find $\nabla\wedge\underline{A}$ at the point (1, −1, 1).

Ans. $3\underline{j}+4\underline{k}$

3.13: If $\underline{A}=x^2y\underline{i} - 2xz\underline{j} + 2yz\underline{k}$, find curl curl $\underline{A}$.

Ans. $(2x+2)\underline{j}$

3.14: Prove $\nabla\wedge(\underline{A}+\underline{B}) = \nabla\wedge\underline{A}+\nabla\wedge\underline{B}$

3.15: If $\underline{A}= 2yz\underline{i} - x^2y\underline{j} + xz^2\underline{k}$, $B= x^2\underline{i}+yz\underline{j} - xy\underline{k}$, and $\phi=2x^2yz^3$, find a) $(\underline{A}\cdot\nabla)\phi$, b) $\underline{A}\cdot\nabla\phi$, c) $(\underline{B}\cdot\nabla)\underline{A}$, d) $(\underline{A}\wedge\nabla)\phi$, e) $\underline{A}\wedge\nabla\phi$.

Ans. a) $8xy^2z^4-2x^4yz^3+6x^3yz^4$ b) $8xy^2z^4-2x^4yz^3+6x^2yz^4$

c) $(2yz^2-2xy^2)\underline{i} - (2x^3y+x^2yz)\underline{j} + (x^2z^2-2x^2yz)\underline{k}$

d) $-(6x^4y^2z^2+2x^3z^5)\underline{i} + (4x^2yz^5-12x^2y^2z^3)\underline{j} + (4x^2yz^4+4x^3y^2z^2)\underline{k}$

e) $\underline{A}\wedge\nabla\phi = (\underline{A}\wedge\nabla)\phi$ (the same as d)

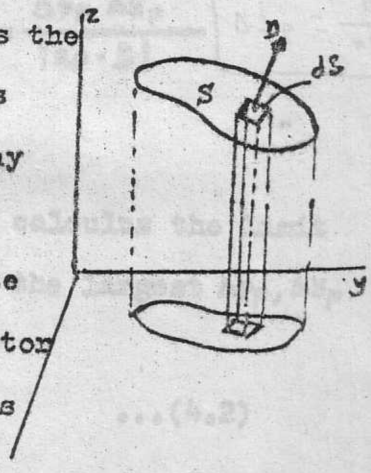
3.16: Prove $\nabla\cdot(\underline{A}\wedge\underline{B}) = \underline{B}\cdot(\nabla\wedge\underline{A}) - \underline{A}\cdot(\nabla\wedge\underline{B})$

$\nabla\wedge(\underline{A}\wedge\underline{B}) = (\underline{B}\cdot\nabla)\underline{A} - \underline{B}(\nabla\cdot\underline{A}) - (\underline{A}\cdot\nabla)\underline{B} + \underline{A}(\nabla\cdot\underline{B})$

$\nabla(\underline{A}\cdot\underline{B}) = (\underline{B}\cdot\nabla)\underline{A} + (\underline{A}\cdot\nabla)\underline{B} + \underline{B}\wedge(\nabla\wedge\underline{A}) + \underline{A}\wedge(\nabla\wedge\underline{B})$

IV-VECTOR INTEGRAL THEOREMS

4.1 Surface Integrals:



Let S be a two-sided surface, as shown in the figure. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\underline{n}$ to any point of the positive side of S is called a positive or outward drawn unit normal. Associate with the differential of surface area dS a vector $d\underline{S}$ whose magnitude is dS and whose direction is that of $\underline{n}$. Then

$$d\underline{S} = \underline{n}\, dS \qquad \ldots(4.1)$$

Subdivide the area S into M elements of area ΔS_p where $p=1,2,3,\ldots,M$. Choose any point P_p within ΔS_p whose coordinates are (x_p, y_p, z_p). Define $\underline{A}(x_p, y_p, z_p) = \underline{A}_p$. Let $\underline{n}_p$ be the positive unit normal to ΔS_p at p. Form the sum

$$\sum_{p=1}^{M} \underline{A}_p \cdot \underline{n}_p \, \Delta S_p$$

where $\underline{A}_p \cdot \underline{n}_p$ is the normal component of $\underline{A}_p$ at P_p.

Now take the limit of this sum as $M \rightarrow \infty$ in such a way that the largest dimension of each ΔS_p approaches zero. This limit, if it exists, is called the surface integral of the normal component of $\underline{A}$ over S and is denoted by

$$\iint_S \underline{A} \cdot \underline{n}\, dS$$

Suppose that the surface S has projection R on the xy plane. The projection of ΔS_p on the xy plane is

$$\left|(\underline{n}_p\,\Delta S_p)\cdot\underline{k}\right| \text{ or } \left|\underline{n}_p\cdot\underline{k}\right|\Delta S_p$$

which is equal to $\Delta x_p\,\Delta y_p$ so that $\Delta S_p = \dfrac{\Delta x_p\,\Delta y_p}{|\underline{n}_p\cdot\underline{k}|}$

$$\boxed{\Delta S_p = \frac{\Delta x_p\,\Delta y_p}{\|\underline{n}_p\cdot\underline{k}\|}}$$

Thus the sum becomes

$$\sum_{p=1}^{M}\underline{A}_p\cdot\underline{n}_p\,\frac{\Delta x_p\,\Delta y_p}{|\underline{n}_p\cdot\underline{k}|}$$

By the fundamental theorem of integral calculus the limit of this sum as $M\rightarrow\infty$ in such a manner that the largest $\Delta x_p, \Delta y_p$ approach zero is

S
look,
3

$$\iint_R \underline{A}\cdot\underline{n}\,\frac{dx\,dy}{|\underline{n}\cdot\underline{k}|} \qquad \ldots(4.2)$$

Similarly, for projection of the surface on yz and zx planes

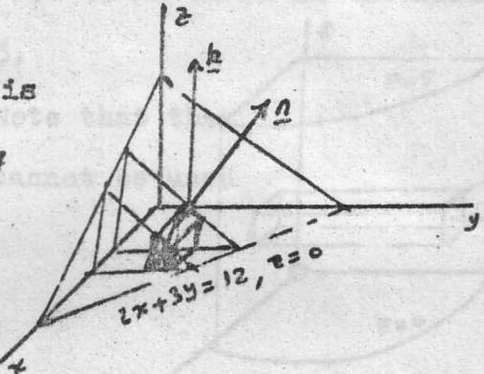
$$\iint_R \underline{A}\cdot\underline{n}\,\frac{dy\,dz}{|\underline{n}\cdot\underline{i}|} \qquad \ldots(4.3)$$

$$\iint_R \underline{A}\cdot\underline{n}\,\frac{dz\,dx}{|\underline{n}\cdot\underline{j}|} \qquad \ldots(4.4)$$

respectively.

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area R of the surface S on one of the coordinate planes.

Ex.4.1:Evaluate $\iint\limits_S \underline{A}\cdot\underline{n}dS$, where $\underline{A} = 18z\underline{i} - 12\underline{j} + 3y\underline{k}$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.



A unit normal to any point of S is

$$\underline{n} = \frac{2\underline{i}+3\underline{j}+6\underline{k}}{\sqrt{2^2+3^2+6^2}} = (2\underline{i}+3\underline{j}+6\underline{k})/7$$

Thus

$$\underline{n}\cdot\underline{k} = 6/7 \quad \text{and} \quad \frac{dxdy}{|\underline{n}\cdot\underline{k}|} = (7/6)dxdy$$

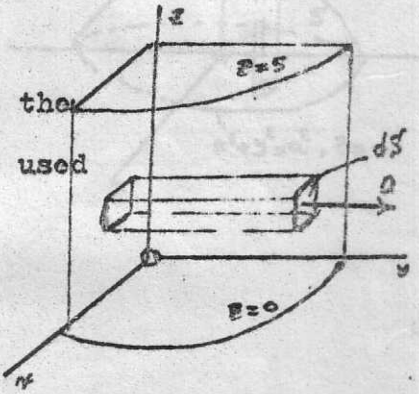
Also $\underline{A}\cdot\underline{n} = \dfrac{36z-36+18y}{7} = (36-12x)/7$, from the equation of S.

$$\text{Then} \iint\limits_S \underline{A}\cdot\underline{n}\,dS = \iint\limits_R \underline{A}\cdot\underline{n}\frac{dxdy}{|\underline{n}\cdot\underline{k}|} = 2\iint\limits_R (3-x)\,dxdy$$

To evaluate this double integral over R, keep x fixed and integrate with respect to y from y=0 to y =(12-2x)/3; then integrate with respect to x from x=0 to x=6. In this manner R is completely covered. The integral becomes

$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)\,dy\,dx = \int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx = 24$$

Ex. 4.2: Evaluate $\iint_S \underline{A}.\underline{n}\,dS$, where $\underline{A} = z\underline{i} + x\underline{j} - 3y^2 z\underline{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.



Project S on the xz plane. Note that the projection of S on the xy plane cannot be used here. Then

$$\iint_S \underline{A}\cdot\underline{n}\,dS = \iint_R \underline{A}\cdot\underline{n}\,\frac{dx\,dz}{|\underline{n}\cdot\underline{j}|}$$

A normal to $x^2+y^2=16$ is $\nabla(x^2+y^2) = 2x\underline{i}+2y\underline{j}$

Thus the unit normal to S is

$$\underline{n} = \frac{2x\underline{i}+2y\underline{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\underline{i}+y\underline{j}}{4}$$

since $x^2+y^2=16$ on S.

$$\underline{A}\cdot\underline{n} = (z\underline{i}+x\underline{j}-3y^2z\underline{k})\cdot(x\underline{i}+y\underline{j})/4 = (xz+xy)/4$$

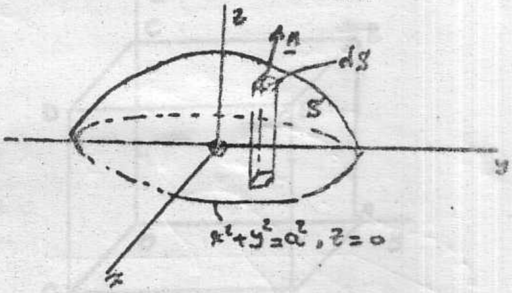
$$\underline{n}\cdot\underline{j} = \frac{(x\underline{i}+y\underline{j})}{4}\cdot\underline{j} = y/4$$

Then the surface integral equals

$$\iint_R \frac{xz+xy}{y}\,dx\,dz = \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dx\,dz$$

$$= \int_{z=0}^{5}(4z+8)\,dz = 90$$

Ex.4.3: If $\underline{F} = y\underline{i} + (x-2xz)\underline{j} - xy\underline{k}$, evaluate $\iint_S (\nabla \wedge \underline{F}) \cdot \underline{n}\, dS$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy plane.



$$\nabla \wedge \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & (x-2xz) & -xy \end{vmatrix} = x\underline{i} + y\underline{j} - 2z\underline{k}$$

A normal to the sphere is

$$\nabla(x^2+y^2+z^2) = 2x\underline{i} + 2y\underline{j} + 2z\underline{k}$$

$$\underline{n} = \frac{2x\underline{i}+2y\underline{j}+2z\underline{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\underline{i}+y\underline{j}+z\underline{k}}{a}$$

The projection of S on the xy plane is the region bounded by the circle $x^2+y^2=a^2$, $z=0$. Then

$$\iint_S (\nabla\wedge\underline{F})\cdot\underline{n}\, dS = \iint_R (\nabla\wedge\underline{F})\cdot\underline{n}\ \frac{dx\, dy}{|\underline{n}\cdot\underline{k}|}$$

$$= \iint_R \frac{x^2+y^2-2z^2}{a}\ \frac{dx\, dy}{z/a}$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\, dy\, dx$$

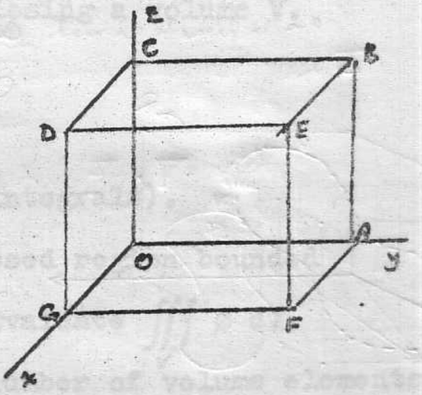
To evaluate the double integral, transform to polar coordinates (ρ,θ) where $x=\rho\cos\theta$, $y=\rho\sin\theta$ and dxdy is replaced by $\rho\, d\rho\, d\theta$. The double integral becomes

$$\int_{\theta=0}^{2\pi}\int_{\rho=0}^{a} \frac{3\rho^2-2a^2}{\sqrt{a^2-\rho^2}}\rho\, d\rho\, d\theta = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{a} \frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\rho\, d\rho\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[(a^2-\rho^2)^{3/2} - a^2\sqrt{a^2-\rho^2} \Big|_{\rho=0}^{a} \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} (a^3-a^3)\, d\theta = 0$$

Ex.4.4: If $\underline{F}=4xz\underline{i}-y^2\underline{j}+yz\underline{k}$, evaluate $\iint_S \underline{F}\cdot\underline{n}dS$, where S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.



Face DEFG, $\underline{n}=\underline{i}$, $x=1$. Then

$$\iint_{DEFG} \underline{F}\cdot\underline{n}\,ds = \int_0^1\int_0^1 (4z\underline{i}-y^2\underline{j}+yz\underline{k})\cdot\underline{i}\,dy\,dz = 2$$

Face ABCO, $\underline{n}=-\underline{i}$, $x=0$. Then

$$\int_0^1\int_0^1 (-y^2\underline{j}+yz\underline{k})\cdot(-\underline{i})\,dy\,dz = 0$$

Face ABEF, $\underline{n}=\underline{j}$, $y=1$. Then

$$\int_0^1\int_0^1 (4xz\underline{i}-\underline{j}+z\underline{k})\cdot\underline{j}\,dx\,dz = -1$$

Face OGDC, $\underline{n}=-\underline{j}$, $y=0$. Then

$$\int_0^1\int_0^1 (4xz\underline{i})\cdot(-\underline{j})\,dx\,dz = 0$$

Face BCDE, $\underline{n}=\underline{k}$, $z=1$. Then

$$\int_0^1\int_0^1 (4x\underline{i}-y^2\underline{j}+y\underline{k})\cdot\underline{k}\,dx\,dy = 1/2$$

Face AFGO, $\underline{n}=-\underline{k}$, $z=0$. Then

$$\int_0^1\int_0^1 (-y^2\underline{j})\cdot(-\underline{k})\,dx\,dy = 0$$

Adding

$$\iint_S \underline{F}\cdot\underline{n}\,dS = 2+0+(-1)+0+1/2+0 = 3/2$$

4.2 Volume Integrals:

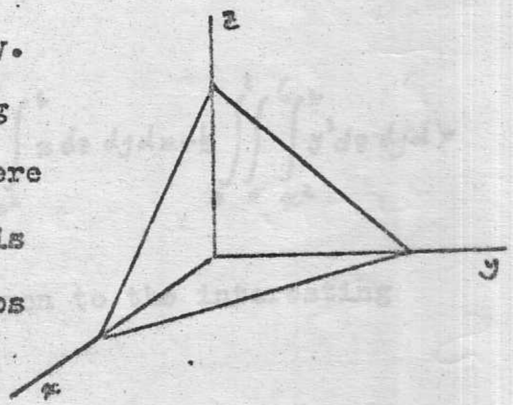
Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \underline{A}\ dV \quad \text{and} \quad \iiint_V \emptyset\ dV$$

are examples of volume integrals (or space integrals).

Ex.4.5: Let $\emptyset = 45x^2y$ and let V denote the closed region bounded by the planes $4x+2y+z=8, x=0, y=0, z=0$. Evaluate $\iiint_V \emptyset\ dV$.

Subdivide the region V into inifinite number of volume elements. Keeping x and y constant, integrate from $z=0$ to $z=8-4x-2y$. Next keep x constant and integrate with respect to y. This amounts to addition of columns having bases in the xy plane ($z=0$) located from where $y=0$ to where $4x+2y=8$ and the integration is from $y=0$ to $y=4-2x$. Finally, we add all slabs parallel to the yz plane, which amounts to integration from $x=0$ to $x=2$. The integration can be written

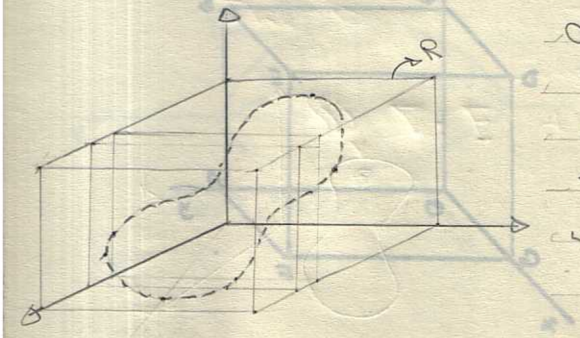


$$\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y\, dz\, dy\, dx = 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)\, dy\, dx$$

$$= 45\int_{x=0}^{2} \frac{1}{3}x^2(4-2x)^3 dx = 128$$

the relation between the divergence and volume integral

notice the relation between the dV, which represent an infinitesimal cube with one of its sides being on the boundary surface of the space which encloses V and the definition of the divergence for which we utilised the idea of an infinitesimal cube.



Cube R containes V ⇒ making planes P_i, P_j, P_k perpendicular to their subscripts ⇒ the distance between each two parallel planes is for example $\delta x, \delta y, \delta z$ which form δV ⇒ $\delta V = \delta x . \delta y . \delta z$

$$\Rightarrow \boxed{V = \lim_{\substack{n \to \infty \\ \delta V \to 0}} \sum_{r=0}^{n=\infty} f_r \, \delta x_r \, \delta y_r \, \delta z_r} \quad ⊛$$

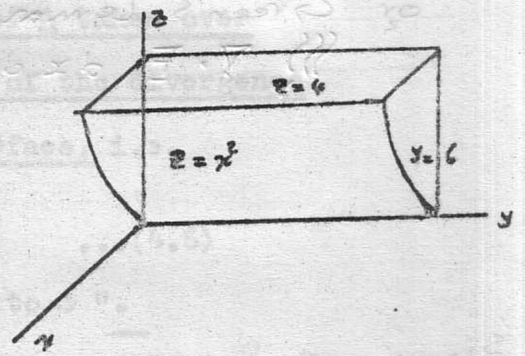
where r represents point P_r

and f is a scalar function defined within and on the boundary of a domain V ⇒ f_r is f at point P_r.

from ⊛ ⇒ $V = \iiint f(x, y, z) \, dx \, dy \, dz$

Ex.4.6:Let $\underline{F}=2xz\underline{i}-x\underline{j}+y^2\underline{k}$.Evaluate $\iiint_V F\,dV$ where V is the region bounded by the surfaces $x=0,y=0,y=6,z=x^2,z=4$.

The region V is covered (a) by keeping x and y fixed and integrating from $z=x^2$ to $z=4$,(b)then by keeping x fixed and integrating from $y=0$ to $y=6$,(c)finally integrating from $x=0$ to $x=2$(Where $z=x^2$ meets $z=4$).



Thus the required integral is

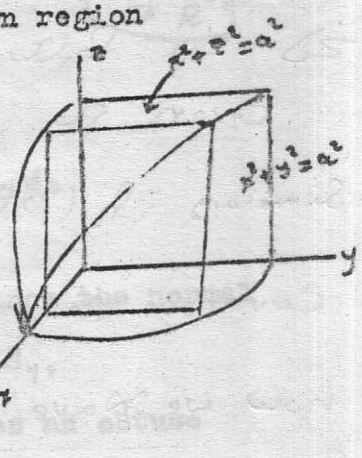
$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}(2xz\underline{i}-x\underline{j}+y^2\underline{k})\,dz\,dy\,dx$$

$$=\underline{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx-\underline{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx+\underline{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx$$

$$=128\underline{i}-24\underline{j}+384\underline{k}$$

Ex.4.7:Find the volume of the region common to the interesting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Required volume= 8 times volume of the shown region



$$=8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}}dz\,dy\,dx$$

$$=8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy\,dx$$

$$=8\int_{x=0}^{a}(a^2-x^2)\,dx=\frac{16a^3}{3}$$

The Divergence Theorem or Gauss's theorem or Green's Lemma.

$$\iiint_V \nabla \cdot \underline{E} \; dx\,dy\,dz = \oiint_S \underline{E} \cdot \underline{n} \; dS$$

Since a function $f(x,y)$ can be represented by a surface s.t $Z = f(x,y)$

notice that this means that we can find find $f_1(x,y)$, $f_2(x,y)$ which can be represented by a surface that covers the given surface S which incloses the volume V

$\Rightarrow$ the $f_1(x,y) = Z$ is a surface that covers the lower part of S

Similary $f_2(x,y) = Z$ is a surface that covers the upper surface of S $\Rightarrow$

now we have covered the whole surface S by f_1 and f_2 $\Rightarrow$ by integrating

$\int_{f_1}^{f_2} \frac{\partial A_3}{\partial z}\, dz$ helps us to get rid of Z as a variable

apply the same process $\boxed{\text{for}}$ $\int_{g_1}^{g_2} \frac{\partial A_2}{\partial y}\, dy$; $\int_{h_1}^{h_2} \frac{\partial A_1}{\partial x}\, dx$

4.3 Gauss Divergence Theorem:

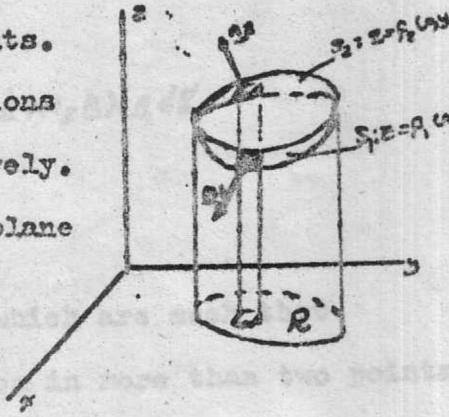
"If V is the volume bounded by a closed surface S and $\underline{A}$ is a vector function of position, with continuous derivatives, then the surface integral of the normal component of $\underline{A}$ taken over the closed surface S is equal to the integral of the divergence of $\underline{A}$ taken over the volume enclosed by the surface, i.e.

$$\iint_S \underline{A}\cdot\underline{n}\, dS = \iiint_V \nabla\cdot\underline{A}\, dV \qquad \ldots(4.6)$$

where $\underline{n}$ is the positive (outward drawn) normal to S ".

Proof:

Let S be a closed surface which is such that any line parallel to the coordinate axes cuts S in at most two points. Assume the equations of the lower and upper portions S_1 and S_2 to be $z=f_1(x,y)$ and $z=f_2(x,y)$ respectively. Denote the projections of the surface on the xy plane by R. Consider



$$\iiint_V \frac{\partial A_3}{\partial z}\, dV = \iiint_V \frac{\partial A_3}{\partial z}\, dz\, dy\, dx = \iint_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z}\, dz\right] dy\, dx$$

$$= \iint_R A_3(x,y,z)\Big|_{z=f_1}^{f_2} dy\, dx = \iint_R \left[A_3(x,y,f_2) - A_3(x,y,f_1)\right] dy\, dx$$

For the upper portion S_2, $dydx = \cos\gamma_2\, dS_2 = \underline{k}\cdot\underline{n}_2\, dS_2$ since the normal $\underline{n}_2$ to S_2 makes an angle γ_2 with $\underline{k}$. For the lower portion S_1, $dydx = -\cos\gamma_1\, dS_1 = -\underline{k}\cdot\underline{n}_1\, dS_1$ since the normal $\underline{n}_1$ to S_1 makes an obtuse angle γ_1 with $\underline{k}$.

Then

$$\iint_R A_3(x,y,f_2)\, dy\, dx = \iint_{S_2} A_3\, \underline{k}\cdot\underline{n}_2\, dS_2$$

$$\iint_R A_3(x,y,f_1)\, dy\, dx = -\iint_{S_1} A_3\, \underline{k}\cdot\underline{n}_1\, dS_1$$

and $$\iint_R A_3(x,y,f_2)\,dy\,dx - \iint_R A_3(x,y,f_1)\,dy\,dx = \iint_{S_2} A_3\,\mathbf{k}\cdot\mathbf{n}_2\,dS_2 + \iint_{S_1} A_3\,\mathbf{k}\cdot\mathbf{n}_1\,dS_1$$

$$= \iint_S A_3\,\mathbf{k}\cdot\mathbf{n}\,dS$$

so that
$$\iiint_V \frac{\partial A_3}{\partial z}\,dV = \iint_S A_3(\mathbf{k}\cdot\mathbf{n})\,dS \qquad (i)$$

Similarly, by projecting S on the other coordinate planes,

$$\iiint_V \frac{\partial A_1}{\partial x}\,dV = \iint_S A_1(\mathbf{i}\cdot\mathbf{n})\,dS \qquad (ii)$$

$$\iiint_V \frac{\partial A_2}{\partial y}\,dV = \iint_S A_2(\mathbf{j}\cdot\mathbf{n})\,dS \qquad (iii)$$

Adding
$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_S (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\mathbf{n}\,dS$$

or
$$\iiint_V (\nabla\cdot\mathbf{A})\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS$$

The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points.

Ex. 4.7: Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ and S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

By the divergence theorem, the required integral is equal to

$$\iiint_V (\nabla\cdot\mathbf{F})\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z-y)\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz\Big|_{z=0}^{1}\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\,dy\,dx = 3/2$$

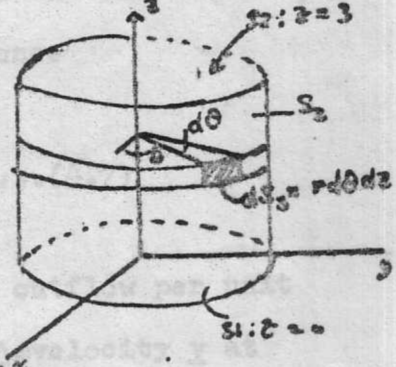
Ex.4.8:Verify the divergence theorem for $\underline{A}=4x\underline{i}-2y^2\underline{j}+z^2\underline{k}$ taken over the region bounded by $x^2+y^2=4, z=0$ and $z=3$.

$$\text{Volume integral}=\iiint_V (\nabla\cdot\underline{A})\,dV=\iiint_V\left[\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}(-2y^2)+\frac{\partial}{\partial z}(z^2)\right]dV$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)\,dz\,dy\,dx=84\pi$$

The surface S of the cylinder consists of a base $S_1(z=0)$, the top $S_2(z=3)$ and the convex portion $S_3(x^2+y^2=4)$. Then

$$\text{Surface integral}=\iint_S \underline{A}\cdot\underline{n}\,dS=\iint_{S_1}\underline{A}\cdot\underline{n}\,dS_1+\iint_{S_2}\underline{A}\cdot\underline{n}\,dS_2+\iint_{S_3}\underline{A}\cdot\underline{n}\,dS_3$$



On $S_1(z=0)$, $\underline{n}=-\underline{k}$, $\underline{A}=4x\underline{i}-2y^2\underline{j}$ & $\underline{A}\cdot\underline{n}=0$

$$\therefore \iint_{S_1}\underline{A}\cdot\underline{n}\,dS_1=0 \qquad (i)$$

On $S_2(z=3)$, $\underline{n}=\underline{k}$, $\underline{A}=4x\underline{i}-2y^2\underline{j}+9\underline{k}$ & $\underline{A}\cdot\underline{n}=9$

$$\therefore \iint_{S_2}\underline{A}\cdot\underline{n}\,dS_2=9\iint_{S_2}dS_2=36\pi, \text{ since } S_2=4\pi \qquad (ii)$$

On S_3 $(x^2+y^2=4)$

$$\underline{n}=\nabla S_3/|\nabla S_3|=\frac{2x\underline{i}+2y\underline{j}}{\sqrt{4x^2+4y^2}}=\frac{x\underline{i}+y\underline{j}}{2}, \text{ since } x^2+y^2=4$$

$$\therefore \underline{A}\cdot\underline{n}=(4x\underline{i}-2y^2\underline{j}+z^2\underline{k})\cdot\left(\frac{x\underline{i}+y\underline{j}}{2}\right)=2x^2-y^3$$

From the figure, $x=2\cos\theta$, $y=2\sin\theta$, $dS_3=2\,d\theta\,dz$ and $r=$

$$\iint_{S_3}\underline{A}\cdot\underline{n}\,dS_3=\int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left[2(2\cos\theta)^2-(2\sin\theta)^3\right]2\,dz\,d\theta=\int_{\theta=0}^{2\pi}48\cos^2\theta\,d\theta=48\pi \qquad (iii)$$

Then the surface integral $=0+36\pi+48\pi=84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Note that evaluation of the surface integral over S_3 could also have been done by projection on the xz or yz coordinate planes.

Definition of the Divergence:

Let div $\underline{A}$ denote the divergence of a vector field at a point p. By the mean-value theorem for integrals, the right side of (4.6) can be written

$$\overline{\operatorname{div} \underline{A}} \iiint_{\Delta V} dV = \overline{\operatorname{div} \underline{A}}\ \Delta V$$

where $\overline{\operatorname{div} \underline{A}}$ is some value intermediate between the maximum and minimum of div $\underline{A}$ throughout ΔV, then

$$\overline{\operatorname{div} \underline{A}} = \frac{\iint_{\Delta S} \underline{A}\cdot\underline{n}\, dS}{\Delta V}$$

Taking the limit as $\Delta V \rightarrow 0$ such that p is always interior to ΔV, $\overline{\operatorname{div} \underline{A}}$ approaches the value of div $\underline{A}$ at point p; hence

$$\operatorname{div} \underline{A} = \lim_{\Delta V \rightarrow 0} \frac{\iint_{\Delta S} \underline{A}\cdot\underline{n}\, dS}{\Delta V} \qquad \ldots(4.7)$$

Physically, div $\underline{A}$ represents the flux or net outflow per unit volume of the vector $\underline{A}$ from the surface Δ S. Let $\underline{A}$=velocity $\underline{v}$ at any point of a moving fluid. Volume of fluid crossing dS in Δt sec.

=volume contained in cylinder of base dS and slant height $\underline{v}\,\Delta t$.

$=(\underline{v}\Delta t)\cdot\underline{n}\ dS = \underline{v}\cdot\underline{n}\ dS\ \Delta t$.

Then, volume per sec. of fluid crossing dS=$\underline{v}\cdot\underline{n}$ dS

Total volume per sec. of fluid emerging from closed surface S=$\iint_S \underline{v}\cdot\underline{n} dS$

From (3.17), $\nabla\cdot\underline{v}$ dV is the volume per sec. of fluid emerging from a volume element dV. The total volume per sec. of fluid emerging from all volume elements is $= \iiint_V \nabla\cdot\underline{v}\ dV$

Thus

$$\iint_S \underline{v}\cdot\underline{n}\ dS = \iiint_V \nabla\cdot\underline{v}\ dV$$

Gauss'Theorem:

Let S be a closed surface and let r denote the position vector of any point (x,y,z) measured from an origin O.Then

$$\iint_S \frac{\underline{r}\cdot\underline{n}\,dS}{r^3} = \iint_S \frac{\underline{n}\cdot\underline{r}}{r^3}dS = \begin{cases} 0, & \text{if O lies outside S} \\ 4\pi, & \text{if O lies inside S} \end{cases} \quad ...(4.8)$$

Proof: By the divergence theorem

$$\iint_S \frac{\underline{n}\cdot\underline{r}}{r^3}dS = \iiint_V \nabla\cdot(\underline{r}/r^3)\,dV = \iiint_V \nabla\cdot(\underline{r}/r^3)\,dV$$

(i) But $\nabla\cdot\underline{r}/r^3 = 0$ (Ex.3.13) everywhere within V provided $r\neq 0$ in V, i.e. provided O is outside of V and thus outside of S.Then

$$\iint_S \frac{\underline{n}\cdot\underline{r}}{r^3}dS = 0$$

(ii) If O is inside S,surround O by a small sphere s of radius a. Let τ denote the region bounded by S and s.Then by the divergence theorem

$$\iint_{S+s = S+s} \frac{\underline{n}\cdot\underline{r}}{r^3}dS = \iint_{S=S} \frac{\underline{n}\cdot\underline{r}}{r^3}dS + \iint_{s=s} \frac{\underline{n}\cdot\underline{r}}{r^3}dS = \iiint_V \nabla\cdot(\underline{r}/r^3)\,dV = 0$$

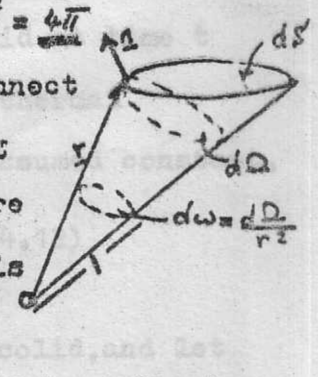
since $r\neq 0$ in τ .Thus

$$\iint_{S=S} \frac{\underline{n}\cdot\underline{r}}{r^3}dS = -\iint_{s=s} \frac{\underline{n}\cdot\underline{r}}{r^3}dS$$

Now on s, r=a, $\underline{n} = -\underline{r}/a$ so that $\frac{\underline{n}\cdot\underline{r}}{r^3} = -\frac{(\underline{r}/a)\cdot\underline{r}}{a^3} = -(\underline{r}\cdot\underline{r})/a^4 = -a^2/a^4 = -1/a^2 = -1/a$

and

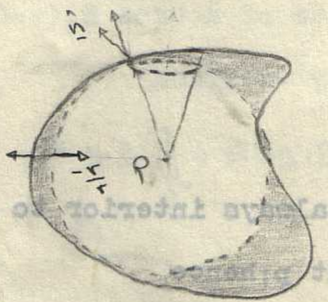
$$\iint_{S=S} \frac{\underline{n}\cdot\underline{r}}{r^3}dS = -\iint_{s=s} \frac{\underline{n}\cdot\underline{r}}{r^3}dS = \iint_s \frac{1}{a^2}dS = \frac{1}{a^2}\iint_s dS = \frac{4\pi a^2}{a^2} = 4\pi \quad \boxed{4\pi}$$

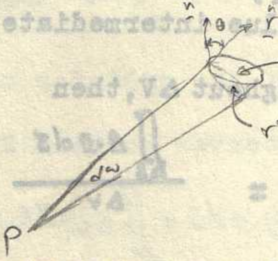


Let dS denote an element of surface area and connect all points on the boundary of dS to O,thereby forming a cone.Let $d\Omega$ be the area of that portion of a sphere with O as centre and radius r which is cut out by this cone.Let $\underline{n}$ be the positive unit normal to dS,then $d\Omega = \pm(\underline{n}\cdot\underline{r})/r\,dS$.The solid angle subtended by dS at O is defined as

$$dw = d\Omega/r^2 = \pm(\underline{n}\cdot\underline{r})/r^3\,dS \quad ...(4.9)$$

and is numerically equal to the area of that portion of a sphere with centre O and unit radius cut out by the cone.





the Shaded region in $\mathbb{R}^3$ is $\hat{S} + S$

$$\nabla \cdot r^{-3}\underline{r} = r^{-3}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)$$

$$+ x\frac{\partial r^{-3}}{\partial x} + y\frac{\partial r^{-3}}{\partial y} + z\frac{\partial r^{-3}}{\partial z} =$$

$$r^{-3} + x\left(-\frac{3}{\cancel{2}}r^{-5} \times \cancel{2}x\right) + y\left(-\frac{3}{\cancel{2}}r^{-5} \times \cancel{2}y\right) + z\left(-\frac{3}{\cancel{2}}r^{-5} \times \cancel{2}z\right)$$

$$3r^{-3} + -3(x^2+y^2+z^2)\,r^{-5} = \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{r^5} = 0$$

Green's Identities:

First identity:

$$\iiint_V [\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi]\,dV = \iint_S (\phi\nabla\psi)\cdot d\underline{S} \qquad \ldots(4.10)$$

Second identity:

$$\iiint_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\underline{S} \qquad \ldots(4.11)$$

Proof: Let $\underline{A} = \phi\nabla\psi$ in the divergence theorem, then

$$\iiint_V \nabla\cdot(\phi\nabla\psi)\,dV = \iint_S \phi\nabla\psi\cdot\underline{n}\,dS = \iint_S (\phi\nabla\psi)\cdot d\underline{S}$$

But $\nabla\cdot(\phi\nabla\psi) = \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$

Thus $$\iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV = \iint_S (\phi\nabla\psi)\cdot d\underline{S} \qquad (i)$$

Interchanging ϕ and ψ

$$\iiint_V [\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)]\,dV = \iint_S (\psi\nabla\phi)\cdot d\underline{S} \qquad (ii)$$

Subtracting (ii) from (i), we have

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\underline{S} \qquad (iii)$$

In the proof we have assumed that ϕ and ψ are scalar functions of position with continuous derivatives of the second order at least.

Heat Equation:

If the temperature at any point (x,y,z) of a solid at time t is U(x,y,z,t) and if K, ρ and c are respectively the thermal conductivity, density and specific heat of the solid, assumed constant, then

$$\frac{\partial U}{\partial t} = k\nabla^2 U \qquad \ldots(4.12)$$

where $k = K/\rho c$

Proof: Let V be an arbitrary volume lying within the solid, and let S denote its surface. The total flux of heat across S, or the quantity of heat leaving S per unit time, is

$$\iint_S (-K\nabla U)\cdot\underline{n}\,dS$$

Thus the quantity of heat entering S per unit time is

$$\iint_S (K\nabla U)\cdot \mathbf{n}\, dS = \iiint_V \nabla\cdot (K\nabla U)\, dV \qquad (i)$$

by the divergence theorem. The heat contained in a volume V is given by $\frac{cal}{gm\,C^\circ}\cdot\frac{gm}{C^3}\cdot C^\circ$

$$\iiint_V c\rho U\, dV \qquad \iiint_V c\rho U\, dV$$

Since density $\rho = \frac{m}{V}$
Specific heat $c = \frac{dQ}{m\,dT}$

Then the time rate of increase of heat is

$$\frac{\partial}{\partial t}\iiint_V c\rho U\, dV = \iiint_V c\rho \frac{\partial U}{\partial t}\, dV = \iiint_V c\rho \frac{\partial U}{\partial t}\, dV \qquad (ii)$$

Equating the right hand sides of (i) and (ii),

$$\iiint_V \left[c\rho\frac{\partial U}{\partial t} - \nabla\cdot(K\nabla U)\right] dV = 0$$

and since V is arbitrary, the integrand, assumed continuous, must be identically zero so that

$$c\rho\frac{\partial U}{\partial t} = \nabla\cdot(K\nabla U)$$

$$\boxed{\frac{\partial U}{\partial t} = \frac{K}{c\rho}\nabla\cdot\nabla U = K\nabla^2 U}$$

or if K, c, ρ are constants

$$\frac{\partial U}{\partial t} = \frac{K}{c\rho}\nabla\cdot\nabla U = k\nabla^2 U \qquad (iii)$$

The quantity k is called the diff$\underset{\sim}{\text{us}}$ivity. For steady-state heat flow (i.e. $\frac{\partial U}{\partial t}=0$), or U is independent of time, the equation reduces to Laplace's equation $\nabla^2 U=0$.

Related Integral Identities:

[illegible])

$$\iiint_V \nabla\varphi\, dV = \iint_S \varphi\mathbf{n}\, dS \qquad \ldots(4.13)$$

Proof: In the divergence theorem, let $\mathbf{A}=\varphi\mathbf{C}$, where $\mathbf{C}$ is a constant vector. Then

$$\iiint_V (\nabla\cdot\varphi\mathbf{C})\, dV = \iint_S \varphi\mathbf{C}\cdot\mathbf{n}\, dS$$

Since

$$\nabla\cdot(\varphi\mathbf{C}) = \nabla\varphi\cdot\mathbf{C} = \mathbf{C}\cdot\nabla\varphi \quad \& \quad \varphi\mathbf{C}\cdot\mathbf{n} = \mathbf{C}\cdot(\varphi\mathbf{n}),$$

Taking $\underline{C}$ outside the integrals,

$$\underline{C}\cdot\iiint_V \nabla\varphi\, dV = \underline{C}\cdot\iint_S \varphi\,\underline{n}\, dS$$

and since $\underline{C}$ is an arbitrary constant vector,

$$\iiint_V \nabla\varphi\, dV = \iint_S \varphi\,\underline{n}\, dS$$

(ii) $$\iiint_V (\nabla\wedge\underline{B})\, dV = \iint_S (\underline{n}\wedge\underline{B})\, dS \qquad \ldots(4.14)$$

Proof:

In the divergence theorem, let $\underline{A}=\underline{B}\wedge\underline{C}$, where $\underline{C}$ is a constant vector. Then

$$\iiint_V \nabla\cdot(\underline{B}\wedge\underline{C})\, dV = \iint_S (\underline{B}\wedge\underline{C})\cdot\underline{n}\, dS$$

Since $\nabla\cdot(\underline{B}\wedge\underline{C}) = \underline{C}\cdot(\nabla\wedge\underline{B})$ & $(\underline{B}\wedge\underline{C})\cdot\underline{n} = \underline{B}\cdot(\underline{C}\wedge\underline{n}) = (\underline{C}\wedge\underline{n})\cdot\underline{B} = \underline{C}\cdot(\underline{n}\wedge\underline{B})$

$$\therefore \iiint_V \underline{C}\cdot(\nabla\wedge\underline{B})\, dV = \iint_S \underline{C}\cdot(\underline{n}\wedge\underline{B})\, dS$$

Taking $\underline{C}$ outside the integrals,

$$\underline{C}\cdot\iiint_V (\nabla\wedge\underline{B})\, dV = \underline{C}\cdot\iint_S (\underline{n}\wedge\underline{B})\, dS$$

and since $\underline{C}$ is an arbitrary constant vector,

$$\iiint_V (\nabla\wedge\underline{B})\, dV = \iint_S (\underline{n}\wedge\underline{B})\, dS$$

(iii) At any point p

$$\nabla\varphi = \lim_{\Delta V\to 0} \frac{\iint_{\Delta S} \varphi\,\underline{n}\, dS}{\Delta V} \qquad \ldots(4.15)$$

$$\nabla\wedge\underline{A} = \lim_{\Delta V\to 0} \frac{\iint_{\Delta S} (\underline{n}\wedge\underline{A})\, dS}{\Delta V} \qquad \ldots(4.16)$$

where ΔV is the volume enclosed by the surface ΔS, and the limit is obtained by shrinking ΔV to the point p.

Proof: From(4.13),

$$\iiint_{\Delta V} \nabla\varphi\, dV = \iint_{\Delta S} \varphi\,\underline{n}\, dS$$

Then

$$\iiint_{\Delta V} \nabla\varphi\cdot\underline{i}\, dV = \iint_{\Delta S} \varphi\,\underline{n}\cdot\underline{i}\, dS$$

Prove that $\iiint_V \nabla \times \underline{B}\; dV = \iint_S (\underline{n} \times \underline{B})\, ds$

Since from the Divergence theorem

$\iiint_V \nabla \cdot \underline{F}\; dV = \iint_S \underline{F} \cdot \underline{n}\; ds$; take $\underline{A}$ a constant vector $\Rightarrow$

Let $\underline{F} = \underline{A} \times \underline{B} \Rightarrow \nabla \cdot \underline{F} = \nabla \cdot (A \times \underline{B})$

$= \boxed{\underline{B} \cdot (\nabla \times \underline{A})} - \underline{A} \cdot (\nabla \times \underline{B}) \Rightarrow$

$$\iiint_V \nabla \cdot (\underline{A} \times \underline{B})\; dV = -\iiint_V A \cdot (\nabla \times \underline{B})\, dV$$

$$= -\underline{A} \cdot \iiint_V \nabla \times \underline{B}\; dV \quad ①$$

$\Rightarrow$ to Complete the proof $\Rightarrow$

$\iint_S (\underline{A} \times \underline{B}) \cdot \underline{n}\, ds = \iiint_V \nabla \cdot (\underline{A} \times \underline{B})\, dV$

but $\underline{n} \cdot (\underline{A} \times \underline{B}) = -\underline{A} \cdot (\underline{n} \times \underline{B}) =$

$\Rightarrow \boxed{\iint_S (\underline{A} \times \underline{B}) \cdot \underline{n}\; ds = -\underline{A} \cdot \iint_S (\underline{n} \times \underline{B})\, dS \quad ②}$

$=$

$-\underline{A} \cdot \iiint \nabla \times \underline{B}\; dV = -\underline{A} \cdot \iint_S (\underline{n} \times \underline{B})\, ds$

$= \iiint_V \nabla \cdot (\underline{A} \times \underline{B})\, dV = \iint_S (\underline{A} \times \underline{B}) \cdot \underline{n}\, ds \quad \square$

Using the same principle employed in deriving (4.7), we have

$$\overline{\nabla\varphi\cdot\underline{i}} = \frac{\iint_{\Delta S}\varphi\underline{n}\cdot\underline{i}\,dS}{\Delta V}$$

Where $\overline{\nabla\varphi\cdot\underline{i}}$ is some value intermediate between the maximum and minimum of $\nabla\varphi\cdot\underline{i}$ throughout Δ V. Taking the limit as $\Delta V \to 0$ in such a way that p is always interior to ΔV, $\overline{\nabla\varphi\cdot\underline{i}}$ approaches the value

$$\nabla\varphi\cdot\underline{i} = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}\varphi\underline{n}\cdot\underline{i}\,dS}{\Delta V} \qquad (i)$$

Similarly we find

$$\nabla\varphi\cdot\underline{j} = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}\varphi\underline{n}\cdot\underline{j}\,dS}{\Delta V} \qquad (ii)$$

$$\nabla\varphi\cdot\underline{k} = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}\varphi\underline{n}\cdot\underline{k}\,dS}{\Delta V} \qquad (iii)$$

Multiplying (i), (ii), (iii) by $\underline{i}, \underline{j}, \underline{k}$ respectively, and adding, using

$$\nabla\varphi = (\nabla\varphi\cdot\underline{i})\,\underline{i} + (\nabla\varphi\cdot\underline{j})\,\underline{j} + (\nabla\varphi\cdot\underline{k})\underline{k}$$

$$\underline{n} = (\underline{n}\cdot\underline{i})\,\underline{i} + (\underline{n}\cdot\underline{j})\,\underline{j} + (\underline{n}\cdot\underline{k})\underline{k}$$

the result (4.15) follows.

Using the same argument, we get, from (4.14)

$$(\nabla\wedge\underline{B})\cdot\underline{i} = \lim_{\Delta V\to 0}\frac{\iint_{\Delta S}(\underline{n}\wedge\underline{B})\cdot\underline{i}\,dS}{\Delta V}$$

and similar results with $\underline{j}$ and $\underline{k}$ replacing $\underline{i}$. Multiplying by $\underline{i}, \underline{j}, \underline{k}$ and adding, the result (4.16) follows.

The results obtained (4.15) and (4.16) can be taken as starting points for definition of gradient and curl respectively.

$$\nabla \circ \square = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} d\underline{S} \circ \square$$

Integral Operator Form for ∘ :

$$\nabla\circ \equiv \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} d\underline{S}\,\circ \qquad \ldots(4.17)$$

where o indicates a dot product, cross product or ordinary product.

If o is the dot product, then for a vector $\underline{A}$

$$\nabla\circ\underline{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} d\underline{S}\circ\underline{A} \qquad \ldots(4.18)$$

or

$$\operatorname{div}\underline{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} d\underline{S}\cdot\underline{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} \underline{A}\cdot\underline{n}\,dS$$

established in (4.7).

Similarly if o is the cross product

$$\operatorname{Curl}\underline{A} = \nabla\wedge\underline{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} d\underline{S}\wedge\underline{A} \qquad \ldots(4.19)$$

$$= \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} (\underline{n}\wedge\underline{A})\,dS$$

established in (4.16).

Also if o is ordinary multiplication, then for a scalar ϕ

$$\nabla\circ\varphi = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} dS\circ\varphi$$

$$\ldots(4.20)$$

or

$$\nabla\varphi = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S} \varphi\,dS$$

established in (4.15).

4.4 Green's Theorem in the Plane:

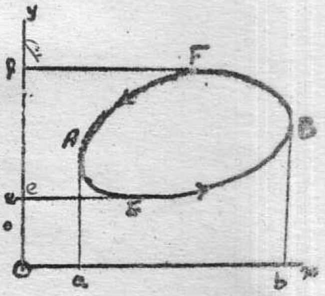
"If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$\underline{F} = (M(x,y), N(x,y))$

$\underline{dR} = (dx, dy)$

$$\underline{F} \cdot \underline{dR} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy \qquad \ldots(4.21)$$

where C is traversed in the positive(counterclockwise) direction"



Proof: Let C be a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points. Let the equations of the curves AEB and AFB be $y=Y_1(x)$ and $y=Y_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dy\right]dx = \int_{x=a}^{b} M(x,y)\Big|_{y=Y_1(x)}^{Y_2(x)} dx = \int_a^b [M(x,Y_2)-M(x,Y_1)]\,dx$$

$$= -\int_a^b M(x,Y_1)\,dx - \int_b^a M(x,Y_2)\,dx = -\oint_C M\,dx$$

Then
$$\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

Similarly let the equations of curves EAF and EBF be $x=X_1(y)$ and $x=X_2(y)$ respectively. Then

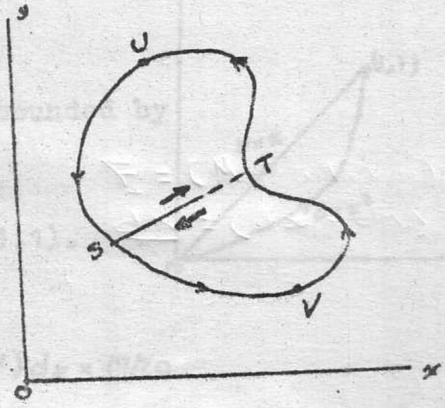
$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy = \int_e^f [N(x_2,y)-N(x_1,y)]\,dy$$

$$= \int_f^e N(x_1,y)\,dy + \int_e^f N(x_2,y)\,dy = \oint_C N\,dy$$

Then
$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$$

Adding (i) and (ii),
$$\oint_C [M\,dx + N\,dy] = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

The above-mentioned proof of Green's theorem in the plane can be extended to the curves C for which lines parallel to the coordinate axes may cut C in more than two points.



Consider a closed curve C such as shown in the figure, in which lines parallel to the axes may meet C in more than two points. By constructing line ST the region is divided into two regions R_1 and R_2 for which Green's theorem applies, i.e.

$$\int_{STUS} M\,dx + N\,dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy \qquad (i)$$

$$\int_{SVTS} M\,dx + N\,dy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy \qquad (ii)$$

Adding the left hand sides of (i) and (ii), we have, omitting the integrand Mdx+Ndy in each case,

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

using the fact that

$$\int_{ST} = - \int_{TS}$$

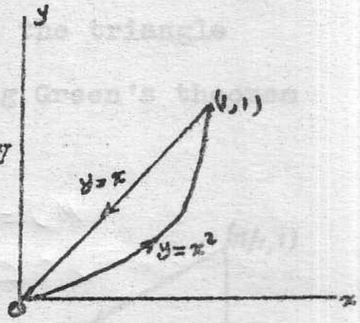
Adding the right hand sides of (i) and (ii), omitting the integrand,

$$\iint_{R_1} + \iint_{R_2} = \iint_{R}$$

where R consists of regions R_1 and R_2. Then

$$\int_{TUSVT} M\,dx + N\,dy = \iint_{R} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$ and the theorem is proved.

A region R, for which any closed curve lying in it can be continuously shrunk to a point without leaving R, is called a simply-connected region. A region which is not simply-connected is called multiply-connected. We have shown that Green's theorem in the plane applies to simply-connected regions bounded by closed curves.



Ex.4.9:Verify Green's theorem in the plane for

$$\oint_C (xy+y^2)\,dx + x^2\,dy$$

where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

$y=x$ and $y=x^2$ intersect at (0,0) and (1,1).

Along $y=x^2$, the line integral equals

$$\int_0^1 [(x)(x^2)+x^4]\,dx + (x^2)(2x)\,dx = \int_0^1 (3x^3+x^4)\,dx = 19/20$$

Along $y=x$ from (1,1) to (0,0) the line integral equals

$$\int_1^0 [(x)(x)+x^2]\,dx + x^2\,dx = \int_1^0 3x^2\,dx = -1$$

Then the required line integral $= 19/20 - 1 = -1/20$

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right]dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x-2y)\,dy\right]dx = \int_0^1 (xy-y^2)\Big|_{x^2}^{x}\,dx$$

$$= \int_0^1 (x^4-x^3)\,dx = -1/20$$

so that the theorem is verified.

Ex.4.10:Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (x\,dy - y\,dx)$. Hence find the area of the ellipse $x=a\cos\theta, y=b\sin\theta$.

In Green's theorem, put $M=-y, N=x$. Then

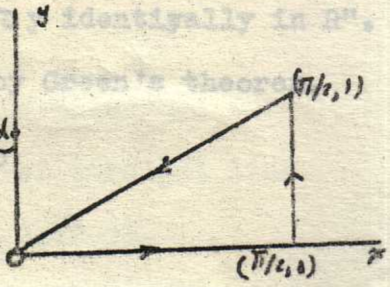
$$\oint_C (x\,dy - y\,dx) = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]dx\,dy = 2\iint_R dx\,dy = 2A$$

where A is the required area. Thus $A = \frac{1}{2}\oint_C (x\,dy - y\,dx)$

For the ellipse, area $= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta+\sin^2\theta)\,d\theta$

$$= \pi ab$$

Ex.4.11: Evaluate $\oint_C (y-\sin x)\,dx + \cos x\,dy$, where C is the triangle of vertices $(0,0)$, $(\pi/2,0)$ and $(\pi/2,1)$; using Green's theorem in the plane.



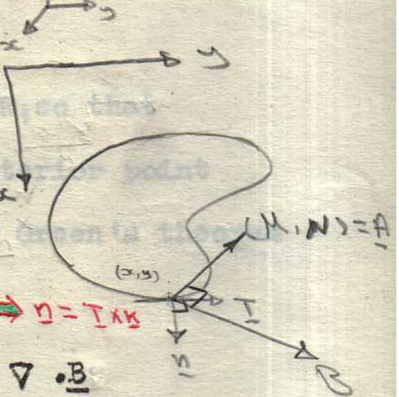
$M=y-\sin x$, $N=\cos x$, $\frac{\partial N}{\partial x}=-\sin x$, $\frac{\partial M}{\partial y}=1$ and

$$\oint_C (M\,dx+N\,dy)=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\iint_R(-\sin x-1)\,dx\,dy$$

$$=\int_{x=0}^{\pi/2}\left[\int_{y=0}^{2x/\pi}(-\sin x-1)\,dy\right]dx$$

$$=\int_0^{\pi/2}\left(-\frac{2x}{\pi}\sin x-\frac{2x}{\pi}\right)dx=-\frac{2}{\pi}-\pi/4$$

Green's Theorem in Space:



Since $M\,dx+N\,dy=\underline{A}\cdot d\underline{r}=\underline{A}\cdot(d\underline{r}/ds)\,ds=\underline{A}\cdot\underline{T}\,ds$,

where $\underline{T}$= unit tangent vector to C, then

$$M\,dx+N\,dy=\underline{A}\cdot(\underline{k}\wedge\underline{n})\,ds,$$

$\underline{T}\perp\underline{k}$ and $\underline{n}$

where $\underline{n}$ is the outward drawn unit normal to C $(\underline{T}=\underline{k}\wedge\underline{n})$. $\Rightarrow$ $\underline{n}=\underline{T}\times\underline{k}$

Since $\underline{A}=M\underline{i}+N\underline{j}$, $\underline{B}=\underline{A}\wedge\underline{k}=(M\underline{i}+N\underline{j})\wedge\underline{k}=N\underline{i}-M\underline{j}$ and $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\nabla\cdot\underline{B}$

Then Green's theorem in the plane becomes

$$\oint_C \underline{B}\cdot\underline{n}\,ds=\iint_R(\nabla\cdot\underline{B})\,dR \qquad \ldots(4.22)$$

$$\oint_C \underline{A}\cdot d\underline{R}=$$
$$\oint_C M\,dx+N\,dy=$$
$$\iint_D\left((\partial N/\partial x)-\partial M/\partial y\right)dx\,dy$$

where $dR=dx\,dy$,

Generalization of this to the case where the differential arc length ds of a closed curve C is replaced by the differential of surface area dS of a closed surface S, and the corresponding plane region R enclosed by C is replaced by the volume V enclosed by S, leads to Gauss' divergence theorem or Green's theorem in space

$$\iint_S(\underline{B}\cdot\underline{n})\,dS=\iiint_V\nabla\cdot\underline{B}\,dV \qquad \ldots(4.23)$$

$$\boxed{\oint_S \underline{F}\cdot\underline{n}\,dS=\iiint_V\nabla\cdot\underline{F}\,dx\,dy\,dz}$$

$\oint_C \underline{A}\cdot d\underline{r} = \oint_C M\,dx + N\,dy = \oint_C \underline{A}\cdot\underline{T}\,ds$ ①

since $\underline{T} = \underline{K}\times\underline{n} \Rightarrow$ ① $= \oint_C \underline{A}\cdot(\underline{K}\times\underline{n})\,ds$ ②

since $\underline{A}\cdot(\underline{K}\times\underline{n}) = -\underline{n}\cdot(\underline{K}\times\underline{A}) = \underline{n}\cdot(\underline{A}\times\underline{K}) \Rightarrow$

② $= \oint_C (\underline{A}\times\underline{K})\cdot\underline{n}\,ds = \oint_C \underline{B}\cdot\underline{n}\,ds$ ③

where $\underline{B} = \underline{A}\times\underline{K} \Rightarrow$ the double integral over the domain $R \Rightarrow$ ③ $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy =$

$\iint_R \nabla\cdot\underline{B}\,dx\,dy$ ④ $\underline{B} = \underline{A}\times\underline{K} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ M & N & 0 \\ 0 & 0 & 1 \end{vmatrix} = N\underline{i} - M\underline{j}$

let $dR = dx\,dy$

$\Rightarrow$ ④ $= \iint_R \nabla\cdot\underline{B}\,dR$. $\nabla\cdot\underline{B} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

* * *

Given S and $\underline{F} \Rightarrow \oiint_S \underline{F}\cdot\underline{n}\,dS$

where $\underline{F}(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$

Independence of the Path:

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous partial derivatives at each point of a simply connected region R.Then "A necessary and sufficient condition that $\oint_C (Mdx+Ndy)=0$ around every closed path C in R is that $\partial N/\partial x=\partial M/\partial y$ identigally in R".

Proof:Sufficiency.Suppose $\partial N/\partial x=\partial M/\partial y$.Then by Green's theorem,

$$\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=0$$

where R is the region bounded by C.

Necessity.Suppose $\oint_C (Mdx+Ndy)=0$ around every closed path C in R and that $\partial N/\partial x\neq\partial M/\partial y$ at some point of R.In particular suppose $\partial M/\partial y-\partial N/\partial x>0$ at the point (x_0,y_0).

By hypothesis $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in R,so that there must be some region τ containing(x_0,y_0) as an interior point for which $\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)>0$.If Γ is the boundary of τ,then by Green's theorem

$$\oint_\Gamma Mdx+Ndy=\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy<0$$

contradicting the hypothesis that $\oint Mdx+Ndy=0$ for all closed curves in R.Thus $\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ cannot be positive.

Similarly we can show that $\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ cannot be negative, and it follows that it must be identically zero,i.e. $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ identically in R.

If $\underline{A}(=M\underline{i}+N\underline{j})$ denotes the force field acting on a particle,then $\oint_C \underline{A}\cdot d\underline{r}$ is the work done in moving the particle around a closed path C and is determined by the value of $\nabla\wedge\underline{A}$.It follows in particular that if $\nabla\wedge\underline{A}=0$ or equivalently if $\underline{A}=\nabla\phi$,then the integral around a closed path is zero.This amounts to saying that the work done in moving the particle from one point in the plane to another is independent of the path in the plane joining the points or that the force field is conservative.

Ex.4.12: Evaluate $\int_{(0,0)}^{(2,1)} (10x^4-2xy^3)\,dx - 3x^2y^2\,dy$ along the path $x^4-6xy^3=4y^2$.

A direct evaluation is difficult. However, noting that

$M = 10x^4 - 2xy^3$, $N = -3x^2y^2$ and $\frac{\partial M}{\partial y} = -6xy^2 = \frac{\partial N}{\partial x}$

it follows that the integral is independent of the path. Then we can use any path, for example the path consisting of straight line segments from (0,0) to (2,0) and then from (2,0) to (2,1).

Along the straight line path from (0,0) to (2,0), y=0, dy=0 and the integral equals

$$\int_{x=0}^{2} 10x^4\,dx = 64$$

Along the straight line path from (2,0) to (2,1), x=2, dx=0 and the integral equals

$$\int_{y=0}^{1} -12y^2\,dy = -4$$

Then the required value of the line integral =64-4=60.

Another Method:

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $(10x^4-2xy^3)dx - 3x^2y^2dy$ is an exact differential of $(2x^5-x^2y^3)$. Then

$$\int_{(0,0)}^{(2,1)} (10x^4-2xy^3)dx - 3x^2y^2dy = \int_{(0,0)}^{(2,1)} d(2x^5-x^2y^3) = 2x^5-x^2y^3\Big|_{(0,0)}^{(2,1)} = 60$$

In this example, let $\underline{A} = (10x^4-2xy^3)\underline{i} - 3x^2y^2\underline{j}$. Since the line integral $\int_{(0,0)}^{(2,1)} \underline{A}\cdot d\underline{r}$ is independent of the path, it follows that the integral around any closed path is zero, then $\nabla\wedge\underline{A}=0$. Therefore $\underline{A} = \nabla\phi$, where $\phi=(2x^5-x^2y^3)$ is called the potential of $\underline{A}$.

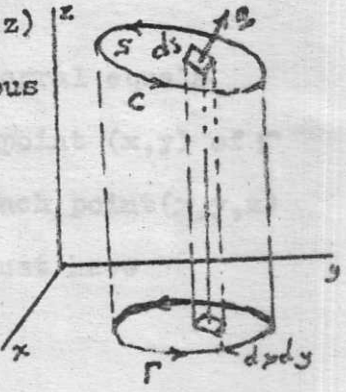
4.5 Stokes'Theorem:

"If S is an open, two-sided surface bounded by a closed, non-intersecting curve C(simple closed curve), ~~and~~ if $\underline{A}$ has continuous derivatives, the line integral of the tangential component of the vector $\underline{A}$ taken around C is equal to the surface integral of the normal component of the curl of $\underline{A}$ taken over S, i.e.

$$\oint_C \underline{A}\cdot d\underline{r} = \iint_S (\nabla\wedge\underline{A})\cdot d\underline{S} \qquad \ldots(4.24)$$

where C is traversed in the positive direction."

Proof: Let S be a surface which is such that its projection on the xy, yz and xz planes are regions bounded by simple closed curves. Assume S to have representation $z=f(x,y)$ or $x=g(y,z)$ or $y=h(x,z)$, where f, g, h are single-valued, continuous and differentiable functions.



Consider first $\iint_S [\nabla\wedge(A_1\underline{i})]\cdot\underline{n}\,dS$

Since
$$\nabla\wedge(A_1\underline{i}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z}\underline{j} - \frac{\partial A_1}{\partial y}\underline{k}$$

Then $[\nabla\wedge(A_1\underline{i})]\cdot\underline{n}\,dS = \left(\frac{\partial A_1}{\partial z}\underline{n}\cdot\underline{j} - \frac{\partial A_1}{\partial y}\underline{n}\cdot\underline{k}\right)dS$ (i)

If $z=f(x,y)$ is taken as the equation of S, then the position vector to any point of S is $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = x\underline{i} + y\underline{j} + f(x,y)\underline{k}$

so that $\frac{\partial \underline{r}}{\partial y} = \underline{j} + \frac{\partial z}{\partial y}\underline{k} = \underline{j} + \frac{\partial f}{\partial y}\underline{k}$

But $\frac{\partial \underline{r}}{\partial y}$ is a vector tangent to S and thus perpendicular to $\underline{n}$, so that

$$\underline{n}\cdot\frac{\partial \underline{r}}{\partial y} = \underline{n}\cdot\underline{j} + \frac{\partial z}{\partial y}(\underline{n}\cdot\underline{k}) = 0 \qquad \text{or} \qquad \underline{n}\cdot\underline{j} = -\frac{\partial z}{\partial y}(\underline{n}\cdot\underline{k})$$

Substitute in (i) to obtain

$$\left(\frac{\partial A_1}{\partial z}(\underline{n}\cdot\underline{j}) - \frac{\partial A_1}{\partial y}(\underline{n}\cdot\underline{k})\right)dS = \left(-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}(\underline{n}\cdot\underline{k}) - \frac{\partial A_1}{\partial y}(\underline{n}\cdot\underline{k})\right)dS$$

or

$$\left[\nabla\wedge(A_1\underline{i})\right]\cdot\underline{n}\,dS = -\left(\frac{\partial A_1}{\partial y}+\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\right)(\underline{n}\cdot\underline{k})\,dS \qquad (ii)$$

Now on S $A_1(x,y,z) = A_1(x,y,f(x,y))$

$$= F(x,y)$$

hence $\dfrac{\partial A_1}{\partial y}+\dfrac{\partial A_1}{\partial z}\dfrac{\partial z}{\partial y} = \dfrac{\partial F}{\partial y}.$

and (ii) becomes

$$\left[\nabla\wedge(A_1\underline{i})\right]\cdot\underline{n}\,dS = -\frac{\partial F}{\partial y}(\underline{n}\cdot\underline{k})\,dS = -\frac{\partial F}{\partial y}\,dx\,dy$$

Then

$$\iint_S\left[\nabla\wedge A_1\underline{i}\right]\cdot\underline{n}\,dS = -\iint_R\frac{\partial F}{\partial y}\,dx\,dy$$

where R is the projection of S on the xy plane.

By Green's theorem for the plane the last integral equals $\oint_\Gamma F\,dx$ where Γ is the boundary of R. Since at each point (x,y) of Γ the value of F is the same as the value of A_1 at each point(x,y,z) of C, and since dx is the same for both curves, we must have

$$\oint_\Gamma F\,dx = \oint_C A_1\,dx$$

or

$$\iint_S\left[\nabla\wedge(A_1\underline{i})\right]\cdot\underline{n}\,dS = \oint_C A_1\,dx$$

Similarly, by projections on the other coordinate planes,

$$\iint_S\left[\nabla\wedge(A_2\underline{j})\right]\cdot\underline{n}\,dS = \oint_C A_2\,dy$$

$$\iint_S\left[\nabla\wedge(A_3\underline{k})\right]\cdot\underline{n}\,dS = \oint_C A_3\,dz$$

Thus by addition

$$\iint_S(\nabla\wedge\underline{A})\cdot\underline{n}\,dS = \oint_C\underline{A}\cdot d\underline{r}$$

Ex.4.13: Verify Stokes'theorem for $A=(2x-y)\underline{i}-yz^2\underline{j}-y^2z\underline{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and centre at the origin. Let $x=\cos t$, $y=\sin t$, $z=0$, $0 \leqslant t < 2\pi$ be parametric equations of C. Then

$$\oint_C \underline{A}\cdot d\underline{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

Also

$$\nabla\wedge\underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & (-y^2z) \end{vmatrix} = \underline{k}$$

Then

$$\iint_S (\nabla\wedge\underline{A})\cdot\underline{n}\,dS = \iint_S \underline{k}\cdot\underline{n}\,dS = \iint_R dx\,dy$$

since $(\underline{n}\cdot\underline{k})dS=dxdy$ and R is the projection of S on the xy plane. This last integral equals

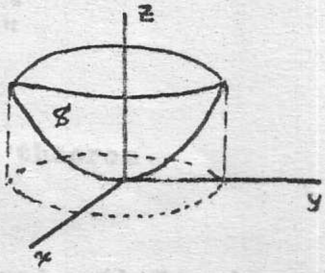
$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

and Stokes'theorem is verified.

Ex.4.14:Verify Stokes'theorem in the particular case

$\underline{A}=x\underline{i}+(2z-x)\underline{j}+y^2\underline{k}$, and let C be the circle $x^2+y^2=4, z=1$.

The parametric equations of the curve C are



x=2 cos t, y= 2 sin t,and z=1,then

$$\oint_C \underline{A}.d\underline{r}=\int_0^{2\pi}\left[-2\sin t+(2-2\cos t)(2\cos t)\right]dt=-4\pi$$

The curve C bounds any number of surfaces,

for example the portion of the paraboliod $4z=x^2+y^2$ beneath the plane z=1.Since on this surface

$$\underline{r}=x\underline{i}+y\underline{j}+\frac{(x^2+y^2)}{4}\underline{k}$$

with

$$\underline{r}_x=\frac{\partial r}{\partial x}=\underline{i}+\frac{x}{2}\underline{k}$$

$$\underline{r}_y=\frac{\partial r}{\partial y}=\underline{j}+\frac{y}{2}\underline{k} \qquad \therefore\ \underline{r}_x\wedge\underline{r}_y=-\frac{x}{2}\underline{i}-\frac{y}{2}\underline{j}+\underline{k}$$

$$\iint_S(\nabla\wedge\underline{A}).d\underline{S}=\iint_R(\nabla\wedge\underline{A}).(\underline{r}_x\wedge\underline{r}_y)\,dx\,dy$$

$$=\iint_R\left[(2y-2)(-x/2)-1\right]dx\,dy=\iint_R\left[x-xy-1\right]dx\,dy$$

$$=\int_{\theta=0}^{2\pi}\int_{\rho=0}^{2}\left[\rho\cos\theta-\rho^2\cos\theta\sin\theta-1\right]\rho\,d\rho\,d\theta=-4\pi$$

Definition of the curl:

If ΔS is a surface bounded by a simple closed curve C,p is any point of ΔS not on C and $\underline{n}$ is a unit normal to ΔS at p,then at p

$$(\text{Curl}\,\underline{A}).\underline{n}=\lim_{\Delta S\to 0}\frac{\oint_C \underline{A}.d\underline{r}}{\Delta S}$$

where the limit is taken in such a way that ΔS shrinks to p.

Proof: By Stokes'theorem

$$\iint_{\Delta S}(\text{Curl}\,\underline{A}).\underline{n}\,dS=\oint_C \underline{A}.d\underline{r}$$

Using the mean value theorem for integrals,this can be written

$$\overline{(\text{Curl}\,\underline{A}).\underline{n}}=\frac{\oint_C \underline{A}.d\underline{r}}{\Delta S}.$$

and the required result follows upon taking the limit as $\Delta S\to 0$.

Irrotational Field Vector: '

"A necessary and sufficient condition that $\oint_C \underline{A}\cdot d\underline{r} = 0$ for every closed curve C is that $\nabla\wedge\underline{A}=0$ identically."

Proof:

Sufficiency. Suppose $\nabla\wedge\underline{A}=0$. Then by Stokes' theorem

$$\oint_C \underline{A}\cdot d\underline{r} = \iint_S (\nabla\wedge\underline{A})\cdot\underline{n}\,dS = 0$$

Necessity. Suppose $\oint_C \underline{A}\cdot d\underline{r}=0$ around every closed path C, and assume $\nabla\wedge\underline{A}\neq 0$ at some point p. Then assuming $\nabla\wedge\underline{A}$ is continuous there will be a region with p as an interior point, where $\nabla\wedge\underline{A}\neq 0$. Let S be a surface contained in this region whose normal $\underline{n}$ at each point has the same direction as $\nabla\wedge\underline{A}$, i.e. $\nabla\wedge\underline{A}=\alpha\underline{n}$ where α is a positive constant. Let C be the boundary of S. Then by Stokes' theorem

$$\oint_C \underline{A}\cdot d\underline{r} = \iint_S (\nabla\wedge\underline{A})\cdot\underline{n}\,dS = \alpha\iint_S (\underline{n}\cdot\underline{n})\,dS > 0$$

which contradicts the hypothesis that $\oint_C \underline{A}\cdot d\underline{r}=0$ and shows that $\nabla\wedge\underline{A}=0$.

It follows that $\nabla\wedge\underline{A}=0$ is also a necessary and sufficient condition for a line integral $\int_{p_1}^{p_2}\underline{A}\cdot d\underline{r}$ to be independent of the path joining points p_1 and p_2. A vector field having this property is called an irrotational field vector.

Ex. 4.15: Consider the vector field $\underline{F}=2xy\underline{i}+(1+x^2)\underline{j}+3z^2\underline{k}$, which has everywhere the property that $\nabla\wedge\underline{F}=0$. Obtain the potential function $\phi(x,y,z)$ such that $\nabla\phi=\underline{F}$.

$$\varphi(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} 2xy\,dx + (1+x^2)\,dy + 3z^2\,dz$$

taken along any convenient path between the end points. In particular a rectangular path can be used, say along C_1, the straight line from (0,0,0) to (x,0,0) then along C_2, the straight line from (x,0,0) to (x,y,0), and finally along C_3 the straight line from (x,y,0) to (x,y,z).

Therefore, $\int_{(0,0,0)}^{(x,y,z)} \underline{F}\cdot d\underline{r} = 0 + (1+x^2)y + z^3$

This function(or this plus any arbitrary constant) is the desired potential function. It can also be obtained by partial integration

$$\int 2xy\,dx = x^2y + k_1(y,z) \qquad = \varphi(x,y,z)$$

$$\int (1+x^2)\,dy = y + x^2y + k_2(x,y) = \varphi(x,y,z)$$

$$\int 3z^2\,dz \quad = z^3 + k_3(x,y,z) \qquad = \varphi(x,y,z)$$

In order that these all be the same function Ø, it is apparent that Ø must be

$$\varphi(x,y,z) = x^2y + y + z^3 + C$$

The line integral between any two points is independent of the path, in this case. For example Ø(0,-1,1) - Ø(1,0,2)=-1+1-8= -8.

Related Integral Identity:

$$\oint d\underline{r}\wedge\underline{B} = \iint(\underline{n}\wedge\nabla)\wedge\underline{B}\,dS \qquad \ldots(4.25)$$

Proof: In Stokes' theorem, let $\underline{A}=\underline{B}\wedge\underline{C}$, where $\underline{C}$ is a constant vector, then

$$\oint d\underline{r}\cdot(\underline{B}\wedge\underline{C}) = \iint_S [\nabla\wedge(\underline{B}\wedge\underline{C})]\cdot\underline{n}\,dS$$

$$\oint \underline{C}\cdot(d\underline{r}\wedge\underline{B}) = \iint_S [(\underline{C}\cdot\nabla)\underline{B} - C(\nabla\cdot\underline{B})]\cdot\underline{n}\,dS$$

$$\underline{C}\cdot\oint d\underline{r}\wedge\underline{B} = \iint_S [(\underline{C}\cdot\nabla)\underline{B}]\cdot\underline{n}\,dS - \iint_S [\underline{C}\,(\nabla\cdot\underline{B})]\cdot\underline{n}\,dS$$

$$= \iint_S \underline{C}\cdot[\nabla(\underline{B}\cdot\underline{n})]\,dS - \iint_S \underline{C}\cdot[\underline{n}(\nabla\cdot\underline{B})]\,dS$$

$$= \underline{C}\cdot\iint_S [\nabla(\underline{B}\cdot\underline{n}) - \underline{n}(\nabla\cdot\underline{B})]\,dS = \underline{C}\cdot\iint_S (\underline{n}\wedge\nabla)\wedge\underline{B}\,dS$$

Problems:IV Vector Integral Theorems

4.1 :Find the surface integral of the vector $\underline{A} = (x^2+y^2)\underline{k}$, taken over the area which is enclosed by the curves $y=x$ and $y=x^2$.

Ans. 3/35

4.2 :Find the surface integral of the field vector

$$\underline{A}= xyz\underline{i} +(y+x)\underline{j} +(y+z)\underline{k},$$

taken over the surface of a cube of side equal to one.

Ans.9/4

4.3 :Calculate the total normal flux of the vector $\underline{r}=x\underline{i}+y\underline{j}+z\underline{k}$, across the closed surfaces of the cube of side 1 parallel to the coordinate axes.

Ans.$3 1^3$

4.4 :Deduce the total normal flux of $\underline{r}=yzx^2\underline{i} + zxy^2\underline{j} +xyz^2\underline{k}$, across the closed sides of a cube of side 1 parallel to the coordinate axes.

Ans.$3\,1^6/4$

4.5 :Evaluate the triple integral $\iiint_R (x^2+y^2+z^2)\,dxdydz$, where R is the 3-dimensional region bounded by $x+y+z=a (a > 0)$, $x=0$, $y=0$ $z=0$.

Ans.$a^5/20$

4.6 :Find the volume of the region R bounded by the parabolic cylinder $z=4-x^2$ and the planes $x=0, y=0, y=6$ and $z=0$.

Ans.32

4.7 :Verify the divergence theorem for $\underline{A}=(2x-z)\underline{i}+x^2y\underline{j}-xz^2\underline{k}$ taken over the region bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

4.8 :Evaluate $\iint_S (xz^2dydz + (x^2y-z^3)dzdx + (2xy+y^2z)dxdy)$, where S is the entire surface of the hemispherical region bounded by $z = a^2-x^2-y^2$ and $z=0$, by using the divergence theorem.

Ans. $2\pi a^5/5$

4.9 :Verify Green's theorem in the plane for the special case $\underline{A}=(x^3-xy)\underline{i} + (x^2-y^2)\underline{j}$ and C is the boundary of the surface S enclosed between the curves $y=x^2$ and $y=2x$.

Ans. 4

4.10 :Verify Green's theorem in the plane for the vector function $\underline{F}=xy^2\underline{i} + (x^2+y)\underline{j}$ where S is the region bounded by the line $x+y=1$ and the coordinate axes.

4.11 :Verify Stokes' theorem for $\underline{A} = 3y\underline{i} - xz\underline{j} + yz^2\underline{k}$, where S is the surface of the paraboloid $2z=x^2+y^2$ bounded by $z=2$ and C is its boundary.

4.12 :Prove that $\underline{F}=(2xz^3+6y)\underline{i} + (6x-2yz)\underline{j} + (3x^2z^2-y^2)\underline{k}$ is a conservative force field. Hence evaluate $\int_C \underline{F}\cdot d\underline{r}$ where C is any path from $(1,-1,1)$ to $(2,1,-1)$.

Ans. 15

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V- CURVILINEAR COORDINATES

(i) 5.1 Orthogonal Curviliear Coordinates:

Let the rectangular coordinates (x,y,z) of any point be expressed as functions of (u_1,u_2,u_3) so that

$$x=x(u_1,u_2,u_3) \quad , \; y=y(u_1,u_2,u_3) \quad , \; z=z(u_1,u_2,u_3) \qquad \ldots(5.1)$$

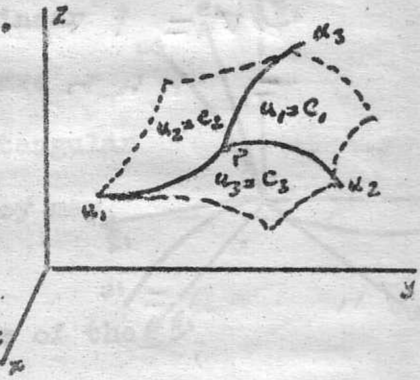
Suppose that (5.1) can be solved for u_1,u_2,u_3 in terms of x,y,z, i.e.

$$u_1=u_1(x,y,z) \quad , \; u_2=u_2(x,y,z) \quad , \; u_3=u_3(x,y,z) \qquad \ldots(5.2)$$

The functions in (5.1) and (5.2) are assumed to be single-valued and to have continuous derivatives so that the correspondence between (x,y,z) and (u_1,u_2,u_3) is unique.

(ii) Given a point p with rectangular coordinates (x,y,z) we can, from (5.2) associate a unique set of coordinates (u_1,u_2,u_3) called the curvilinear coordinates of p. The sets of equations (5.1) or (5.2) define a transformation of coordinates.

The surfaces $u_1=c_1, u_2=c_2, u_3=c_3$, where c_1,c_2,c_3 are constants, are called coordinate surfaces and each pair of these surfaces intersect in curves called coordinate curves or lines. If the coordinate surfaces intersect at right angles the curvilinear coordinate system is called orthogonal. The u_1, u_2 and u_3 coordinate curves of a curvilinear system are analogous to the x, y, and z coordinate axes of a rectangular system.

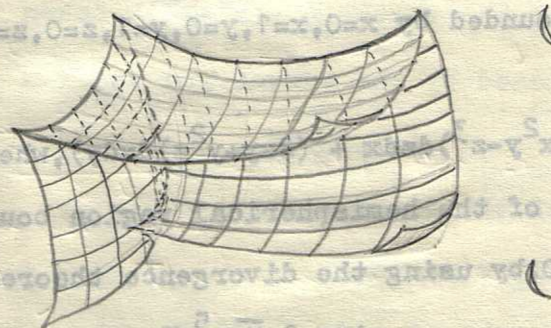


5.2 Unit Vestors in Curvilinear Systems:

Let $\underline{r}=x\underline{i}+y\underline{j}+z\underline{k}$ be the position vector of a point p. Then (5.1) can be written $\underline{r}=\underline{r}(u_1,u_2,u_3)$. A tangent vector to the u_1 curve at p (for which u_2 and u_3 are constants) is $\frac{\partial r}{\partial u_1}$. Then a unit tangent ector in this direction is $\underline{e}_1=\frac{\partial r}{\partial u_1}/|\frac{\partial r}{\partial u_1}|$ so that $\frac{\partial r}{\partial u_1}=h_1\underline{e}_1$ where $h_1=|\frac{\partial r}{\partial u_1}|$

$$\underline{F}(\underline{r}) = (F_1(\underline{r}), F_2(\underline{r}), F_3(\underline{r})) \quad \text{where } \boxed{\underline{r} = (x, y, z)}$$

let $\underline{r} = (x, y, z) = (x(u_1, u_2, u_3), y(u_1, u_2, u_3), z(u_1, u_2, u_3))$



$\underline{F}(u_1, u_2, u_3) = (x, y, z) =$

$\Big(x = x(u_1, u_2, u_3),$
$y = y(u_1, u_2, u_3),$
$z = z(u_1, u_2, u_3) \Big)$ and

$\underline{F}(x, y, z) =$

$\Big(u_1 = u_1(x, y, z),$
$u_2 = u_2(x, y, z),$
$u_3 = u_3(x, y, z) \Big)$

or a scalar valued function
$f(u_1, u_2, u_3)$.

Look!

[1] For safty $\boxed{\nabla u_i(x,y,z)}$ is used when we deal with the surface of u_i, while $\boxed{\dfrac{\partial \underline{r}(u_1, u_2, u_3)}{\partial u_i}}$ is used when we deal with the coordinate curve along which u_i varies.

∇u_i is parallel to $\partial \underline{r}(u_1, u_2, u_3) / \partial u_i$ having the same direction, that of increase (max.) of u_i

$\nabla u_i = \dfrac{\partial u_i}{\partial n} \underline{n}$

$\Rightarrow$

$\nabla u_i \cdot \underline{n} = \dfrac{\partial u}{\partial n}$

$\Rightarrow$ here $\underline{n}$ corrispond to $\underline{e}_i = \dfrac{\left(\dfrac{\partial \underline{r}(u_1, u_2, u_3)}{\partial u_i}\right)}{\left\| \dfrac{\partial \underline{r}(u_1, u_2, u_3)}{\partial u_i} \right\|}$

here we use the coordinate curve $\Rightarrow$ the tangent to it is also normal to, notice here, the surface which corrispond to the same const. coordinate

i.e. ∇u_i is the normal and

$\dfrac{\partial \underline{r}(u_1, u_2, u_3)}{\partial u_i}$ is the tangent to the coordinate curve. To sum Go To [1]

$\Rightarrow$ if $\underline{e}_i = \dfrac{\nabla u_i}{\| \nabla u_i \|}$

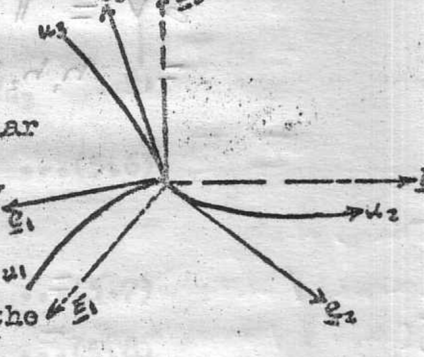
$$\underline{e}_i = \dfrac{\left(\dfrac{\partial \underline{r}(u_1, u_2, u_3)}{\partial u_i}\right)}{\left\| \left(\dfrac{\partial \underline{r}(u_1, u_2, u_3)}{\partial u_i}\right) \right\|}$$

Which is a unit vector tangent to the Coordinate Curve of u_i. $i = 1, 2, 3$

Similarly, if $\underline{e}_2$ and $\underline{e}_3$ are unit tangent vectors to the u_2 and u_3 curves at p respectively, then $\frac{\partial r}{\partial u_2} = h_2\, \underline{e}_2$ and $\frac{\partial r}{\partial u_3} = h_3\, \underline{e}_3$ where $h_2 = \left|\frac{\partial r}{\partial u_2}\right|$ & $h_3 = \left|\frac{\partial r}{\partial u_3}\right|$ .The quantities h_1, h_2, h_3 are called scale factos. The unit vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are in the directions of increasing u_1, u_2, u_3, respectively.

(ii) Since ∇u_1 is a vector at p normal to the surface $u_1 = c_1$, a unit vector in this direction is given by $\underline{E}_1 = \nabla u_1 / |\nabla u_1|$ .Similarly, the unit vectors $\underline{E}_2 = \nabla u_2 / |\nabla u_2|$ and $\underline{E}_3 = \nabla u_3 / |\nabla u_3|$ at p are normal to the surfaces $u_2 = c_2$ and $u_3 = c_3$ respectively.

Thus at each point p of a curviliear system there exist, in general, two sets of unit vectors, $\underline{e}_1, \underline{e}_2, \underline{e}_3$ tangent to the coordinate curves and $\underline{E}_1, \underline{E}_2, \underline{E}_3$ normal to the coordinate surfaces. The sets become identical if and only if the curvilinear coordinate system is orthogonal. Both sets are analogous to the $\underline{i}, \underline{j}, \underline{k}$ unit vectors in rectangular coordinates but are unlike them in that they may change direction from point to point.



A vector $\underline{A}$ can be represented in terms of the unit base vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ or $\underline{E}_1, \underline{E}_2, \underline{E}_3$ in the form

$$\underline{A} = A_1 \underline{e}_1 + A_2 \underline{e}_2 + A_3 \underline{e}_3 \qquad \ldots(5.3)$$

or

$$\underline{A} = a_1 \underline{E}_1 + a_2 \underline{E}_2 + a_3 \underline{E}_3 \qquad \ldots(5.4)$$

where A_1, A_2, A_3 and a_1, a_2, a_3 are the respective components of A in each system.

$$\underline{R}(u_1,u_2,u_3) = \left(x(u_1,u_2,u_3),\ y(u_1,u_2,u_3),\ z(u_1,u_2,u_3)\right)$$

$$\boxed{\frac{\partial \underline{r}(u_1,u_2,u_3)}{\partial u_i} = \left\| \frac{\partial \underline{r}(u_1,u_2,u_3)}{\partial u_i} \right\| e_i \qquad i = 1,2,3}$$

$$\frac{\partial \underline{r}(u_1,u_2,u_3)}{\partial u_i} = \| \partial \underline{r} / \partial u_i \| \ \underline{e}_i = h_i \underline{e}_i$$

where $\underline{e}_i$ points in the direction of increase of u_i

and h_i the scale factor which is defined to be the rate at which the arc length increases on the ith coordinate curve, with respect to u_i,

$$\underline{R}(u_1,u_2,u_3) = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} =$$

$$x(u_1,u_2,u_3)\,\underline{i} + y(u_1,u_2,u_3)\,\underline{j} + z(u_1,u_2,u_3)\,\underline{k}$$

$$\frac{\partial \underline{R}}{\partial u_i} = \frac{\partial x}{\partial u_i}\underline{i} + \frac{\partial y}{\partial u_i}\underline{j} + \frac{\partial z}{\partial u_i}\underline{k}$$

(5)

5.3 Arc length and Volume Elements:

(i) From $\underline{r}=\underline{r}(u_1,u_2,u_3)$ we have

$$d\underline{r}=\frac{\partial r}{\partial u_1}du_1+\frac{\partial r}{\partial u_2}du_2+\frac{\partial r}{\partial u_2}du_3 \quad \text{in general, by the Pythagoream theorem}$$

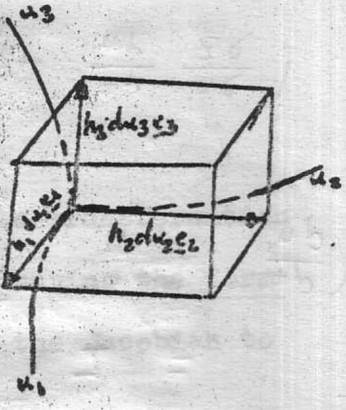
$$=h_1du_1\underline{e}_1+h_2du_2\underline{e}_2+h_3du_3\underline{e}_3 \qquad \ldots(5.5)$$

Then the differential of arc length ds is determined from $ds^2=d\underline{r}.d\underline{r}$. For orthogonal systems,

$$\underline{e}_1\cdot\underline{e}_2=\underline{e}_2\cdot\underline{e}_3=\underline{e}_3\cdot\underline{e}_1=0$$

and

$$ds^2=h_1^2du_1^2+h_2^2du_2^2+h_3^2du_3^2$$



(ii) Along a u_1 curve, u_2 and u_3 are constants so that $d\underline{r}=h_1du_1\underline{e}_1$. Then the differential of arc length ds_1 along u_1 at p is h_1du_1. Similarly the differential arc lengths along u_2 and u_3 at p are $ds_2=h_2du_2$, $ds_3=h_3du_3$

The area elements are given by

$$dA_1=\left|(h_2du_2\underline{e}_2)\wedge(h_3du_3\underline{e}_3)\right|=h_2h_3\left|\underline{e}_2\wedge\underline{e}_3\right|du_2du_3$$

$$=h_2h_3\,du_2du_3 \qquad \ldots(5.6)$$

since $|\underline{e}_2\wedge\underline{e}_3|=|\underline{e}_1|=1$. Similarly

$$dA_2=h_1h_3du_1du_3 \qquad \ldots(5.7)$$

$$dA_3=h_1h_2du_1du_2 \qquad \ldots(5.8)$$

(iii) The volume element for an orthogonal curvilinear coordinate system is given by

$$dV=h_1h_2h_3du_1du_2du_3 \qquad \ldots(5.9)$$

since $\underline{e}_1\cdot(\underline{e}_2\wedge\underline{e}_3)=1$.

$d\underline{R} = \frac{\partial \underline{R}}{\partial u_1} du_1 + \frac{\partial \underline{R}}{\partial u_2} du_2 + \frac{\partial \underline{R}}{\partial u_3} du_3$ (in general)

Since $\frac{\partial \underline{R}}{\partial u_i} = h_i \underline{e}_i \Rightarrow d\underline{R} = \sum_{i=1}^{3} \frac{\partial \underline{R}}{\partial u_i} du_i = \sum_{i=1}^{3} \left\| \frac{\partial \underline{R}}{\partial u_i} \right\| du_i \underline{e}_i$

the Proof is easy if we use Pythagorous theorem, displacement along each $\underline{e}_i$, $i = 1,2,3$ $\Rightarrow d\underline{R} = \sum_{i=1}^{3} h_i du_i \underline{e}_i$ [1]

$\| d\underline{R} \| = ds$ [1]

the arc length along each Coordinate Curve

$ds_i = \| h_i du_i \underline{e}_i \|$ [2]

$\Rightarrow$ the area element on some Coordinate surface $\underline{e}_i$ and $\underline{e}_j$ and $i \neq j$ is $\underline{e}_i \times \underline{e}_j = \underline{e}_k$ and $i \neq j \neq k$

$$d\underline{S}_{ij} = h_i du_i \underline{e}_i \times h_j du_j \underline{e}_j = h_i h_j du_i du_j \underline{e}_k$$

$$\| d\underline{S}_{ij} \| = h_i h_j du_i du_j \quad [3]$$

$\Rightarrow$ the Volume element generated is

$$dV = \| h_i du_i \underline{e}_i \cdot (h_j du_j \underline{e}_j \times h_k du_k \underline{e}_k) \| \quad [4]$$

$$= h_1 h_2 h_3 du_1 du_2 du_3$$

(b) 5.4 The Jacobian of Transformation:

Eq.(5.9) can be written as

$$dV = \left| (h_1 du_1 \underline{e}_1) \cdot [(h_2 du_2 \underline{e}_2) \wedge (h_3 du_3 \underline{e}_3)] \right|$$

$$\overset{\circledast}{=} \left| \frac{\partial \underline{r}}{\partial u_1} \cdot \left(\frac{\partial \underline{r}}{\partial u_2} \wedge \frac{\partial \underline{r}}{\partial u_3} \right) \right| du_1 du_2 du_3 = \left| \frac{\partial (x,y,z)}{\partial (u_1,u_2,u_3)} \right| du_1 du_2 du_3$$

where
$$\frac{\partial (x,y,z)}{\partial (u_1,u_2,u_3)} = J\left(\frac{(x,y,z)}{(u_1,u_2,u_3)}\right) = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} = h_1 h_2 h_3 \quad \ldots(5.10)$$

is called the Jacobian of the transformation.

If the Jacobian equals zero identically then

$$\frac{\partial \underline{r}}{\partial u_1}, \frac{\partial \underline{r}}{\partial u_2}, \frac{\partial \underline{r}}{\partial u_3}$$

Go to ⊛

look

are coplanar vectors and the curvilinear coordinate transformation breaks down, i.e there is a relation between x,y,z having the form F(x,y,z)=0 identically. We shall therefore require the Jacobian to be different from zero.

Since

$$\nabla u_1 \cdot (\nabla u_2 \wedge \nabla u_3) = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_2}{\partial z} \end{vmatrix} = J\left(\frac{u_1, u_2, u_3}{x, y, z}\right) \quad \ldots(5.11)$$

Then

$$\left\{ \frac{\partial \underline{r}}{\partial u_1} \cdot \left(\frac{\partial \underline{r}}{\partial u_2} \wedge \frac{\partial \underline{r}}{\partial u_3} \right) \right\} \left\{ \nabla u_1 \cdot (\nabla u_2 \wedge \nabla u_3) \right\} = 1 \quad \ldots(5.12)$$

or

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) J\left(\frac{u_1, u_2, u_3}{x, y, z}\right) = 1 \quad \ldots(5.13)$$

given $E(u_1, u_2, u_3) = (x, y, z) =$

$(x(u_1,u_2,u_3),\ y(u_1,u_2,u_3),\ z(u_1,u_2,u_3)) \Rightarrow$

$$\Rightarrow \frac{dE(u_1,u_2,u_3)}{dt} = \frac{dx}{dt}e_1 + \frac{dy}{dt}e_2 + \frac{dz}{dt}e_3 =$$

$$\left(\frac{\partial x}{\partial u_1}\frac{du_1}{dt} + \frac{\partial x}{\partial u_2}\frac{du_2}{dt} + \frac{\partial x}{\partial u_3}\frac{du_3}{dt}\right) e_1 +$$

$$\left(\frac{\partial y}{\partial u_1}\frac{du_1}{dt} + \frac{\partial y}{\partial u_2}\frac{du_2}{dt} + \frac{\partial z}{\partial u_3}\frac{du_3}{dt}\right) e_2 +$$

$$\left(\frac{\partial z}{\partial u_1}\frac{du_1}{dt} + \frac{\partial z}{\partial u_2}\frac{du_2}{dt} + \frac{\partial z}{\partial u_3}\frac{du_3}{dt}\right) e_3 =$$

$$\left[\left(\frac{du_1}{dt} \quad \frac{du_2}{dt} \quad \frac{du_3}{dt}\right) \cdot \begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{bmatrix}\right] \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

The Jacobian transformation

$\partial(x,y,z)/\partial(u_1,u_2,u_3)) =$

$$J((x,y,z)/(u_1,u_2,u_3)) = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} =$$

$h_1 h_2 h_3 \quad \Rightarrow$

$$J((x,y,z)/(u_1,u_2,u_3)) \cdot J((u_1,u_2,u_3)/(x,y,z)) = \frac{h_1 h_2 h_3}{h_1 h_2 h_3} = 1$$

The Inverse Jacobian transformation

$(\nabla u_1 \cdot (\nabla u_2 \times \nabla u_3)) =$

$$J((u_1,u_2,u_3)/(x,y,z)) = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{vmatrix} = \frac{1}{h_1 h_2 h_3}$$

$= \|\nabla u_1\| \, \|\nabla u_2 \times \nabla u_3\| \cos(0)$

$\|\nabla u_1\| \, \|\nabla u_2\| \|\nabla u_3\| \sin \pi/2 =$

$\|\nabla u_1\| \|\nabla u_2\| \|\nabla u_3\|$; but $\underline{\|\nabla u_i\|}$

$= \underline{du_i / ds_i}$ Go To [2]

$= \underline{1/h_i}$, where $h_i = \frac{ds_i}{du_i} = \left\|\frac{\partial \mathbf{R}}{\partial u_i}\right\|$

i.e. using [1]

Ex.5.1: If u_1, u_2, u_3 are general coordinates, show that

$$\frac{\partial r}{\partial u_1}, \frac{\partial r}{\partial u_2}, \frac{\partial r}{\partial u_3} \quad \& \quad \nabla u_1, \nabla u_2, \nabla u_3$$

are reciprocal systems of vectors.

We must show that $\dfrac{\partial r}{\partial u_p} \cdot \nabla u_q = \begin{cases} 1 & \text{if } p=q \\ 0 & \text{if } p \neq q \end{cases}$ where p and q can have any of the values 1,2,3.

We have $\quad d\underline{r} = \dfrac{\partial r}{\partial u_1} du_1 + \dfrac{\partial r}{\partial u_2} du_2 + \dfrac{\partial r}{\partial u_3} du_3$

Multiply by $\nabla u_1 \cdot$. Then

$$\nabla u_1 \cdot d\underline{r} = du_1 = \left(\nabla u_1 \cdot \frac{\partial r}{\partial u_1}\right) du_1 + \left(\nabla u_1 \cdot \frac{\partial r}{\partial u_2}\right) du_2 + \left(\nabla u_1 \cdot \frac{\partial r}{\partial u_3}\right) du_3$$

or $\quad \nabla u_1 \cdot \dfrac{\partial r}{\partial u_1} = 1 \;, \; \nabla u_1 \cdot \dfrac{\partial r}{\partial u_2} = 0 \;, \; \nabla u_1 \cdot \dfrac{\partial r}{\partial u_3} = 0$

Similarly, upon multiplying by $\nabla u_2 \cdot$ and $\nabla u_3 \cdot$, the remaining relations are proved.

Ex.5.2: Show that the square of the element of arc length in general curvilinear coordinates can be expressed by

$$ds^2 = \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, du_p\, du_q$$

We have

$$d\underline{r} = \frac{\partial r}{\partial u_1} du_1 + \frac{\partial r}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3 = \underline{\alpha}_1 du_1 + \underline{\alpha}_2 du_2 + \underline{\alpha}_3 du_3$$

Then

$$\begin{aligned} ds^2 = d\underline{r} \cdot d\underline{r} &= \underline{\alpha}_1 \cdot \underline{\alpha}_1 du_1^2 + \underline{\alpha}_1 \cdot \underline{\alpha}_2 du_1 du_2 + \underline{\alpha}_1 \cdot \underline{\alpha}_3 du_1 du_3 \\ &\quad + \underline{\alpha}_2 \cdot \underline{\alpha}_2 du_2^2 + \underline{\alpha}_2 \cdot \underline{\alpha}_1 du_2 du_1 + \underline{\alpha}_2 \cdot \underline{\alpha}_3 du_2 du_3 \\ &\quad + \underline{\alpha}_3 \cdot \underline{\alpha}_3 du_3^2 + \underline{\alpha}_3 \cdot \underline{\alpha}_1 du_3 du_1 + \underline{\alpha}_3 \cdot \underline{\alpha}_2 du_3 du_2 \\ &= \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, du_p\, du_q \quad \text{where } g_{pq} = \underline{\alpha}_p \cdot \underline{\alpha}_q \end{aligned}$$

This is called the fundamental quadratic form or metric form.

The quantities g_{pq} are called metric coefficients and are symmetric, i.e. $g_{pq} = g_{qp}$. If $g_{pq} = 0, p \neq q$, then the coordinate system is orthogonal. In this case

$$g_{11} = h_1^2 \quad , \; g_{22} = h_2^2 \quad , \; g_{33} = h_3^2$$

Since $d\underline{R} = \sum_{i=1}^{3} h_i \, du_i \, \underline{e}_i$ and $\| d\underline{R} \| = ds$

$\Rightarrow (ds)^2 = d\underline{R} \cdot d\underline{R} = \sum_{i=1}^{3} h_i \, du_i \, \underline{e}_i \cdot (h_1 \, du_1 \, \underline{e}_1 + h_2 \, du_2 \, \underline{e}_2 + h_3 \, du_3 \, \underline{e}_3)$

$= \sum_{i=1}^{3} h_i \, du_i \, \underline{e}_i \cdot h_1 \, du_1 \, \underline{e}_1 + \sum_{i=1}^{3} h_i \, du_i \, \underline{e}_i \cdot h_2 \, du_2 \, \underline{e}_2 +$

$\sum_{i=1}^{3} h_i \, du_i \, \underline{e}_i \cdot h_3 \, du_3 \, \underline{e}_3 =$

$\sum_{i=1}^{3} \sum_{j=1}^{3} h_i \, du_i \, \underline{e}_i \cdot h_j \, du_j \, \underline{e}_j =$ where when $i \neq j \Rightarrow \underline{e}_i \cdot \underline{e}_j = 0$

$\sum_{i=1}^{3} \sum_{j=1}^{3} h_i h_j \, du_i \, du_j \, (\underline{e}_i \cdot \underline{e}_j) =$

$(ds)^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij} \, du_i \, du_j$ which is

where $g_{ij} = 0$ when $i \neq j$ the metric form

Ex.5.3: If $F(x,y,u,v)=0$ and $G(x,y,u,v)=0$, find u_x, u_y, v_x, v_y.

The two equations in general define the dependent variables u and v as (implicit) functions of the independent variables x and y. Using the subscript notation, we have

$$dF = F_x dx + F_y dy + F_u du + F_v dv = 0 \qquad (i)$$

$$dG = G_x dx + G_y dy + G_u du + G_v dv = 0 \qquad (ii)$$

Also, since u and v are functions of x and y,

$$du = u_x dx + u_y dy \qquad (iii) \quad \text{and} \quad dv = v_x dx + v_y dy \qquad (iv)$$

Substituting (iii) and (iv) in (i) and (ii) yields

$$dF = (F_x + F_u u_x + F_v v_x)dx + (F_y + F_u u_y + F_v v_y)dy = 0 \qquad (v)$$

$$dG = (G_x + G_u u_x + G_v v_x)dx + (G_y + G_u u_y + G_v v_y)dy = 0 \qquad (vi)$$

Since x and y are independent, the coefficients of dx and dy in (v) and (vi) are zero. Hence we obtain

$$\left.\begin{aligned} F_u u_x + F_v v_x &= -F_x \\ G_u u_x + G_v v_x &= -G_x \end{aligned}\right\} (vii) \quad \& \quad \left.\begin{aligned} F_u u_y + F_v v_y &= -F_y \\ G_u u_y + G_v v_y &= -G_y \end{aligned}\right\} (viii)$$

Solving (vii) and (viii) gives

$$u_x = \frac{\partial u}{\partial x} = \begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix} \Bigg/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = -\frac{\partial(F,G)}{\partial(x,v)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

$$u_y = \frac{\partial u}{\partial y} = \begin{vmatrix} -F_y & F_v \\ -G_y & G_v \end{vmatrix} \Bigg/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = -\frac{\partial(F,G)}{\partial(y,v)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

$$v_x = \frac{\partial v}{\partial x} = -\frac{\partial(F,G)}{\partial(u,x)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

$$v_y = \frac{\partial v}{\partial y} = -\frac{\partial(F,G)}{\partial(u,y)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

The functional determinant $\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$ denoted by $\frac{\partial(F,G)}{\partial(u,v)}$ or $J\left(\frac{F,G}{u,v}\right)$ is the Jacobian of F and G with respect to u and v and is supposed $\neq 0$.

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial u}\,du + \frac{\partial F}{\partial v}\,dv = 0 \qquad (1)$$

Since $u(x,y) \Rightarrow du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$ (2)

$$dG = \frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy + \frac{\partial G}{\partial u}\,du + \frac{\partial G}{\partial v}\,dv = 0 \qquad (3)$$

Since $v(x,y) \Rightarrow dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$ (4)

from (2),(4) in (1) $\Rightarrow$ $dF = (F_x + F_u u_x + F_v v_x)\,dx + (F_y + F_u u_y + F_v v_y)\,dy = 0$

Since dx and dy are independent variable $\Rightarrow$

dx and $dy \neq 0$ $\Rightarrow$

$F_u u_x + F_v v_x = -F_x$ ① and $F_u u_y + F_v v_y = -F_y$ ②

Similary $G_u u_x + G_v v_x = -G_x$ ③ and $G_u u_y + G_v v_y = -G_y$ ④

$\Rightarrow$ from ① and ③, also from ② and ④

$$\frac{u_x}{-\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}} = \frac{v_x}{-\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}} = = \frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \Rightarrow$$

$u_x = -J((F,G)/(x,v)) \,/\, J((F,G)/(u,v))$ ①

$v_x = -J((F,G)/(u,x)) \,/\, J((F,G)/(u,v))$ ②

$u_y = -J((F,G)/(y,v)) \,/\, J((F,G)/(u,v))$ ③

$v_y = -J((F,G)/(u,y)) \,/\, J((F,G)/(u,v))$ ④

5.4 Gradient, Divergence, Curl & Laplacian in Orthogonal Coordinates:

Gradient:

Let $\nabla\phi = f_1\underline{e}_1 + f_2\underline{e}_2 + f_3\underline{e}_3$ where f_1, f_2, f_3 are to be determined.

Since $d\underline{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3$

$$= h_1 \underline{e}_1 du_1 + h_2 \underline{e}_2 du_2 + h_3 \underline{e}_3 du_3$$

We have

$$d\varphi = \nabla\varphi \cdot d\underline{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3 \qquad (i)$$

But
$$d\varphi = \frac{\partial\varphi}{\partial u_1} du_1 + \frac{\partial\varphi}{\partial u_2} du_2 + \frac{\partial\varphi}{\partial u_3} du_3 \qquad (ii)$$

Equating (i) and (ii),

$$f_1 = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}, \quad f_2 = \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}, \quad f_3 = \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}$$

Then

$$\nabla\varphi = \frac{\underline{e}_1}{h_1}\frac{\partial\varphi}{\partial u_1} + \frac{\underline{e}_2}{h_2}\frac{\partial\varphi}{\partial u_2} + \frac{\underline{e}_3}{h_3}\frac{\partial\varphi}{\partial u_3} \qquad \ldots(5.14)$$

This indicates the operator equivalence

$$\nabla \equiv \frac{\underline{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\underline{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\underline{e}_3}{h_3}\frac{\partial}{\partial u_3} \qquad \ldots(5.15)$$

which reduces to the usual expression for the operator ∇ in rectangular coordinates.

Let $\phi = u_1$ in (5.14). Then

$\nabla u_1 = \frac{\underline{e}_1}{h_1}$ & $|\nabla u_1| = |\underline{e}_1|/h_1 = 1/h_1$ i.e. $\nabla u_1 = \left(\frac{1}{h_1}, 0, 0\right)$

Similarly

$|\nabla u_2| = 1/h_2$ & $|\nabla u_3| = 1/h_3$

Then $\nabla u_2 \wedge \nabla u_3 = \frac{\underline{e}_2 \wedge \underline{e}_3}{h_2 h_3} = \frac{\underline{e}_1}{h_2 h_3}$ and $\underline{e}_1 = h_2 h_3 (\nabla u_2 \wedge \nabla u_3)$

Similarly

$$\underline{e}_2 = h_3 h_1 (\nabla u_3 \wedge \nabla u_1) \quad \& \quad \underline{e}_3 = h_1 h_2 (\nabla u_1 \wedge \nabla u_2) \qquad \ldots(5.16)$$

$\nabla f = (\underline{e}_1 \cdot \nabla f)\,\underline{e}_1 + (e_2 \cdot \nabla f)\,\underline{e}_2 + (\underline{e}_3 \cdot \nabla f)\,\underline{e}_3$

Since $\nabla \varphi = \frac{\partial \varphi}{\partial n}\,\underline{n} \Rightarrow \nabla \varphi \cdot d\underline{R}$; $d\underline{R} = \sum_{i=1}^{3} h_i\, du_i\, \underline{e}_i$

$\Rightarrow \left\| \frac{\partial \varphi}{\partial n} \right\| \| d\underline{R} \| \cos\theta$ when the angle (θ) between $\| d\underline{R} \|$ and $\underline{n}$ is zero $\Rightarrow$ $\underset{\partial n = |d\underline{R}|}{}$ $\nabla \varphi \cdot d\underline{R} = d\varphi = \frac{\partial \varphi}{\partial u_1} du_1 + \frac{\partial \varphi}{\partial u_2} du_2 + \frac{\partial \varphi}{\partial u_3} du_3 =$

if $\nabla \varphi = \sum_{i=1}^{3} f_i\, \underline{e}_i \Rightarrow \nabla \varphi \cdot d\underline{R} = d\phi =$

i.e $\frac{\partial \varphi}{\partial n} \| d\underline{R} \| \cos\theta$ but $ds = \| d\underline{R} \| \Rightarrow ds \cos\theta = dn$

$= \sum_{i=1}^{3} f_i\, h_i\, du_i = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial u_i} du_i$

$\Rightarrow f_i = \frac{1}{h_i} \frac{\partial \varphi}{\partial u_i}$; $i = 1, 2, 3 \Rightarrow \nabla \varphi = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial \varphi}{\partial u_i} \underline{e}_i$

Now if $\varphi = u_i \Rightarrow \| \nabla u_i \| = \| (\frac{1}{h_i})\, \underline{e}_i \| = 1/h_i$

$\Rightarrow \nabla u_2 \times \nabla u_3 = \frac{\underline{e}_2 \times \underline{e}_3}{h_2 h_3} = \frac{\underline{e}_1}{h_2 h_3} \Rightarrow$

$\underline{e}_1 = h_2 h_3 (\nabla u_2 \times \nabla u_3)$ i.e,

$e_i = h_j h_k (\nabla u_j \times \nabla u_k)$ and $i \neq j \neq k$

$$\boxed{\nabla \square \equiv \frac{\underline{e}_1}{h_1} \frac{\partial \square}{\partial u_1} + \frac{\underline{e}_2}{h_2} \frac{\partial \square}{\partial u_2} + \frac{\underline{e}_3}{h_3} \frac{\partial \square}{\partial u_3}}$$

$$\boxed{\nabla \square \equiv \sum_{i=1}^{3} \frac{\underline{e}_i}{h_i} \frac{\partial \square}{\partial u_i}}$$

Divergence:

Since

$$\overset{①}{\nabla\cdot}(A_1\underline{e}_1) = \nabla\cdot(A_1h_2h_3\,\nabla u_2\wedge\nabla u_3) \quad \text{, from (5.16)}$$

$$\overset{②}{=} \nabla(A_1h_2h_3)\cdot(\nabla u_2\wedge\nabla u_3) + A_1h_2h_3\,\overset{③}{\underline{\nabla\cdot(\nabla u_2\wedge\nabla u_3)}}$$

$$= \nabla(A_1h_2h_3)\cdot\left(\frac{\underline{e}_2}{h_2}\wedge\frac{\underline{e}_3}{h_3}\right)+0$$

$$= \left[\frac{\underline{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\underline{e}_2}{h_2}\frac{\partial}{\partial u_2}(A_1h_2h_3)+\frac{\underline{e}_3}{h_3}\frac{\partial}{\partial u_3}(A_1h_2h_3)\right]\cdot\frac{\underline{e}_1}{h_2h_3}$$

$$= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(A_1h_2h_3)$$

Then

$$\nabla\cdot\underline{A} = \nabla\cdot(A_1\underline{e}_1+A_2\underline{e}_2+A_3\underline{e}_3)$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\partial}{\partial u_2}(A_2h_3h_1)+\frac{\partial}{\partial u_3}(A_3h_1h_2)\right] \quad \text{——— (5.17)}$$

Curl:

Since

$$\nabla\wedge\underline{A} = \nabla\wedge[A_1\underline{e}_1+A_2\underline{e}_2+A_3\underline{e}_3]$$

$$\nabla\wedge(A_1\underline{e}_1) = \nabla\wedge(A_1h_1\nabla u_1) = \nabla(A_1h_1)\wedge\nabla u_1 + \underline{A_1h_1\nabla\wedge\nabla u_1}$$

$$= \nabla(A_1h_1)\wedge\frac{\underline{e}_1}{h_1} = \left[\frac{\underline{e}_1}{h_1}\frac{\partial}{\partial u_1}(A_1h_1)+\frac{\underline{e}_2}{h_2}\frac{\partial}{\partial u_2}(A_2h_2)+\frac{\underline{e}_3}{h_3}\frac{\partial}{\partial u_3}(A_3h_3)\right]\wedge\frac{\underline{e}_1}{h_1}$$

$$= \underline{\frac{\underline{e}_2}{h_3h_1}\frac{\partial}{\partial u_3}(A_1h_1)} - \underline{\frac{\underline{e}_3}{h_1h_2}\frac{\partial}{\partial u_2}(A_1h_1)}$$

$$\& \quad \nabla\wedge(A_2\underline{e}_2) = \underline{\frac{\underline{e}_3}{h_1h_2}\frac{\partial}{\partial u_1}(A_2h_2)} - \underline{\frac{\underline{e}_1}{h_2h_3}\frac{\partial}{\partial u_3}(A_2h_2)}$$

$$\nabla\wedge(A_3\underline{e}_3) = \underline{\frac{\underline{e}_1}{h_2h_3}\frac{\partial}{\partial u_2}(A_3h_3)} - \underline{\frac{\underline{e}_2}{h_3h_1}\frac{\partial}{\partial u_1}(A_3h_3)}$$

$$\therefore \quad \nabla\wedge\underline{A} = \frac{\underline{e}_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3)-\frac{\partial}{\partial u_3}(A_2h_2)\right]+\frac{\underline{e}_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(A_1h_1)-\frac{\partial}{\partial u_1}(A_3h_3)\right]$$

$$+\frac{\underline{e}_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(A_2h_2)-\frac{\partial}{\partial u_2}(A_1h_1)\right]$$

$$\nabla \cdot \phi \underline{A} = (\nabla \phi) \cdot \underline{A} + \phi (\nabla \cdot \underline{A})$$

$$\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot \nabla \times \underline{A} - \underline{A} \cdot \nabla \times \underline{B}$$

$$\nabla \times \nabla \phi = 0$$ irrotational field

$$\nabla \cdot (\nabla \times \underline{A}) = 0$$ Solenoidal

1) why ∇ didn't opperate directly on $A_i e_i$ in ① !

2) to reach ② ∇ must opperate directly on the opperand to the right of it.

3) by ③ why $\nabla \times \nabla \phi = 0$ in orthogonal curvilinear coordinates.

$$\nabla \cdot \underline{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_2 h_1) \right]$$

$$\nabla \times \phi \underline{A} = \nabla \phi \times \underline{A} + \phi (\nabla \times \underline{A})$$

$$\nabla \times \underline{A} = \frac{e_1}{h_2 h_3} \left[\frac{\partial (A_3 h_3)}{\partial u_2} - \frac{\partial (A_2 h_2)}{\partial u_3} \right] + \frac{e_2}{h_3 h_1} \left[\frac{\partial (A_1 h_1)}{\partial u_3} - \frac{\partial (A_3 h_3)}{\partial u_1} \right] + \frac{e_3}{h_1 h_2} \left[\frac{\partial (A_2 h_2)}{\partial u_1} - \frac{\partial (A_1 h_1)}{\partial u_2} \right]$$

This can be written

$$\nabla \wedge \underline{A} = \begin{vmatrix} h_1\underline{e}_1 & h_2\underline{e}_2 & h_3\underline{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix} \Big/ (h_1h_2h_3) \qquad \ldots(5.18)$$

Laplacian:

From (5.14)

$$\nabla\psi = \frac{\underline{e}_1}{h_1}\frac{\partial\psi}{\partial u_1} + \frac{\underline{e}_2}{h_2}\frac{\partial\psi}{\partial u_2} + \frac{\underline{e}_3}{h_3}\frac{\partial\psi}{\partial u_3}$$

If $\underline{A} = \nabla\psi$, then

$$A_1 = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}, \quad A_2 = \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}, \quad A_3 = \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}$$

Then

$$\nabla\cdot\underline{A} = \nabla\cdot\nabla\psi = \nabla^2\psi$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right] \qquad (5.19)$$

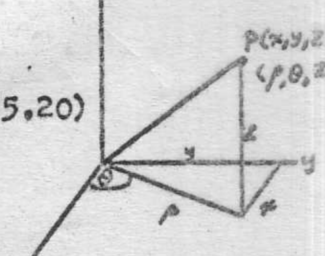
Eqs.(5.14),(5.17),(5.18) and(5.19) reduce to the usual expressions in rectangular coordinates if we replace (u_1,u_2,u_3) by (x,y,z) in which case $\underline{e}_1,\underline{e}_2,\underline{e}_3$ are replaced by $\underline{i},\underline{j},\underline{k}$ and $h_1=h_2=h_3=1$.

5.5:Special Curvilinear Coordinates:

Cylindrical Coordinates(,θ,z):

Transformation equations:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z \qquad \ldots(5.20)$$



where $\rho \geq 0$, $0 \leq \theta < 2\pi$, $-\infty < z < \infty$

Scale factors: $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$

Element of arc length: $ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$ $\qquad \ldots(5.21)$

Jacobian: $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \rho$

Element of volume: $dV = \rho\, d\rho\, d\theta\, dz$

Note that corresponding results can be obtained for polar coordinates in the plane by omitting z dependence, while the element of volume is replaced by the element of area, $dA = \rho\, d\rho\, d\theta$.

The position vector of any point in cylindrical coordinates is $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = \rho\cos\theta\,\underline{i} + \rho\sin\theta\,\underline{j} + z\underline{k}$

The tangent vectors to the ρ, θ and z curves are given respectively by

$$\frac{\partial \underline{r}}{\partial \rho} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \quad , \quad \frac{\partial \underline{r}}{\partial \theta} = -\rho\sin\theta\,\underline{i} + \rho\cos\theta\,\underline{j} \quad , \quad \frac{\partial \underline{r}}{\partial z} = \underline{k}$$

The unit vectors in these directions are

$$\underline{e}_1 = \frac{\partial \underline{r}/\partial \rho}{|\partial \underline{r}/\partial \rho|} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$$

$$\underline{e}_2 = \frac{\partial \underline{r}/\partial \theta}{|\partial \underline{r}/\partial \theta|} = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}$$

$$\underline{e}_3 = \frac{\partial \underline{r}/\partial z}{|\partial \underline{r}/\partial z|} = \underline{k}$$

Then

$$\underline{e}_1 \cdot \underline{e}_2 = 0$$

$$\underline{e}_2 \cdot \underline{e}_3 = 0$$

$$\underline{e}_3 \cdot \underline{e}_1 = 0$$

and so $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are mutually perpendicular and the coordinate system is orthogonal.

Also, for cylindrical coordinates (ρ, θ, z),

$$u_1 = \rho \ , \ u_2 = \theta \ , \ u_3 = z$$

Then

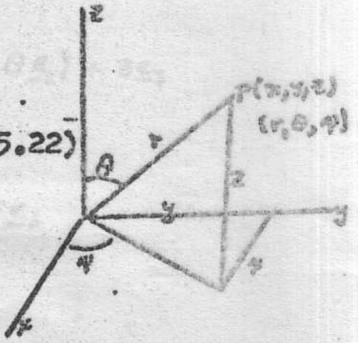
$$\nabla\varphi = \underline{e}_1 \frac{\partial\varphi}{\partial\rho} + \frac{\underline{e}_2}{\rho}\frac{\partial\varphi}{\partial\theta} + \underline{e}_3 \frac{\partial\varphi}{\partial z}$$

$$\nabla\cdot\underline{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(A_1\rho) + \frac{\partial}{\partial\theta}(A_2) + \frac{\partial}{\partial z}(A_3\rho)\right]$$

$$\nabla\wedge\underline{A} = \frac{\underline{e}_1}{\rho}\left[\frac{\partial}{\partial\theta}(A_3) - \frac{\partial}{\partial z}(A_2\rho)\right] + \underline{e}_2\left[\frac{\partial}{\partial z}(A_1) - \frac{\partial}{\partial\rho}(A_3)\right] + \frac{\underline{e}_3}{\rho}\left[\frac{\partial}{\partial\rho}(A_2\rho) - \frac{\partial}{\partial\theta}(A_1)\right]$$

$$\nabla^2\varphi = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial\varphi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial\varphi}{\partial z}\right)\right]$$

Spherical Coordinates(r,θ,ϕ):



Transformation equations:

$x = r\sin\theta\cos\phi \;,\; y = r\sin\theta\sin\phi \;,\; z = r\cos\theta$ ...(5.22)

where $r \geq 0 \;,\; 0 \leq \theta \leq \pi \;,\; 0 \leq \phi < 2\pi$

Scale factors $h_1 = 1,\; h_2 = r \;,\; h_3 = r\sin\theta$

Element of arc length: $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$

Jacobian: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ ...(5.23)

Element of volume: $dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

Here $u_1 = r \;,\; u_2 = \theta \;,\; u_3 = \phi$

Then

$$\nabla\varphi = \underline{e}_1\frac{\partial\varphi}{\partial r} + \frac{\underline{e}_2}{r}\frac{\partial\varphi}{\partial\theta} + \frac{\underline{e}_3}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}$$

$$\nabla\cdot\underline{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(A_1 r^2\sin\theta) + \frac{\partial}{\partial\theta}(A_2 r\sin\theta) + \frac{\partial}{\partial\phi}(A_3 r)\right]$$

$$\nabla\wedge\underline{A} = \frac{\underline{e}_1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(A_3 r\sin\theta) - \frac{\partial}{\partial\phi}(A_2 r)\right] + \frac{\underline{e}_2}{r\sin\theta}\left[\frac{\partial}{\partial\phi}(A_1) - \frac{\partial}{\partial r}(A_3 r\sin\theta)\right]$$
$$+ \frac{\underline{e}_3}{r}\left[\frac{\partial}{\partial r}(A_2 r) - \frac{\partial}{\partial\theta}(A_1)\right]$$

$$\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\varphi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\varphi}{\partial\phi}\right)\right]$$

Ex.5.4: Express the velocity $\underline{v}$ and acceleration $\underline{a}$ of a particle in cylindrical coordinates.

In rectangular coordinates the position vector $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, and the velocity and acceleration vectors are

$$\underline{v} = \frac{d\underline{r}}{dt} = \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k} \quad \& \quad \underline{a} = \frac{d^2\underline{r}}{dt^2} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

In cylindrical coordinates

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = (\rho\cos\theta)(\cos\theta\,\underline{e}_1 - \sin\theta\,\underline{e}_2)$$
$$+ (\rho\sin\theta)(\sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2) + z\underline{e}_3$$

Then

$$= \rho\underline{e}_1 + z\underline{e}_3$$

$$\underline{V} = \frac{d\underline{r}}{dt} = \dot{\rho}\,\underline{e}_1 + \rho\frac{d\underline{e}_1}{dt} + \underline{e}_3\frac{dz}{dt} = \dot{\rho}\,\underline{e}_1 + \rho\dot{\theta}\,\underline{e}_2 + \dot{z}\,\underline{e}_3$$

Differentiating again,

$$\underline{a} = \frac{d^2\underline{r}}{dt^2} = \frac{d}{dt}(\dot{\rho}\,\underline{e}_1 + \rho\dot{\theta}\,\underline{e}_2 + \dot{z}\,\underline{e}_3)$$

Since

$$\frac{d\underline{e}_1}{dt} = \frac{d}{dt}(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) \quad \& \quad \frac{d\underline{e}_2}{dt} = \frac{d}{dt}(-\sin\theta\,\underline{i} + \cos\theta\,\underline{j})$$

Then

$$\frac{d\underline{e}_1}{dt} = \dot{\theta}\,\underline{e}_2 \qquad \& \qquad \frac{d\underline{e}_2}{dt} = -\dot{\theta}\,\underline{e}_1$$

Thus

$$\underline{a} = \dot{\rho}\frac{d\underline{e}_1}{dt} + \ddot{\rho}\,\underline{e}_1 + \rho\dot{\theta}\frac{d\underline{e}_2}{dt} + \rho\ddot{\theta}\,\underline{e}_2 + \dot{\rho}\dot{\theta}\,\underline{e}_2 + \ddot{z}\,\underline{e}_3$$

or $$\underline{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\underline{e}_1 + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\underline{e}_2 + \ddot{z}\,\underline{e}_3$$

Ex.5.5: Using Gauss divergence theorem, evaluate

$$\iint[xz^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2xy + y^2z)\,dz\,dy]$$

where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$; change to spherical coordinates for the evaluation.

The integral can be written as $\int_S \underline{A}\cdot d\underline{S}$, where

$$\underline{A} = xz^2\,\underline{i} + (x^2y - z^3)\,\underline{j} + (2xy + y^2z)\underline{k}$$

Then by the divergence theorem, the integral equals

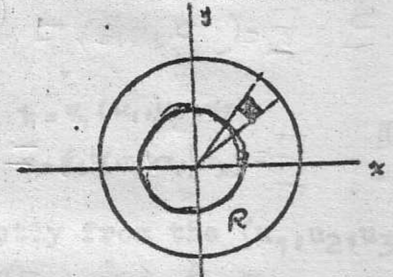
$$\iiint_V \nabla\cdot\underline{A}\;dV = \iiint_V (x^2 + y^2 + z^2)\,dV$$

but the evaluation, although possible, is tedious. It is easier to use spherical coordinates. Since $x^2 + y^2 + z^2 = r^2$ and the volume element is $r^2\sin\theta\,dr\,d\theta\,d\psi$, the integral is

$$4\int_{\psi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2\cdot r^2\sin\theta\,dr\,d\theta\,d\psi = 2\pi a^5/5$$

Ex.5.6: Evaluate $\iint_R \sqrt{x^2+y^2}\, dxdy$, where R is the region in the xy-plane bounded by $x^2+y^2=4$ and $x^2+y^2=9$.

The presence of x^2+y^2 suggests the use of polar coordinates (ρ, θ) where $x=\rho\cos\theta$, $y=\rho\sin\theta$. Under this transformation the region R is mapped into the region R'.



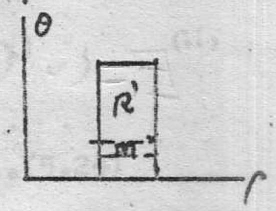
Since

$$\frac{\partial(x,y)}{\partial(\rho,\theta)}=\rho$$

it follows that

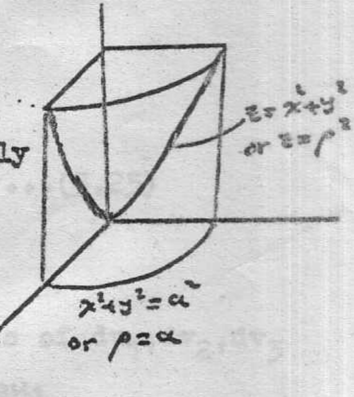
$$\iint_R \sqrt{x^2+y^2}\, dx\, dy = \iint_{R'} \sqrt{x^2+y^2}\left|\frac{\partial(x,y)}{\partial(\rho,\theta)}\right| d\rho\, d\theta = \iint_{R'} \rho\cdot\rho\, d\rho\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\rho=2}^{3} \rho^2\, d\rho\, d\theta = \frac{38\pi}{3}$$



Ex.5.7: Find the volume of the region above the xy plane bounded by the paraboloid $z=x^2+y^2$ and the cylinder $x^2+y^2=a^2$.

The volume is found by using cylindrical coordinates. In these coordinates the equations for the paraboloid and the cylinder are respectively $z=\rho^2$ and $\rho=a$. Then



Required volume

$$= 4\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{a}\int_{z=0}^{\rho^2} \rho\, dz\, d\rho\, d\theta$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{a} \rho^3\, d\rho\, d\theta = \pi/2\ a^4$$

5.6 General Coordinates:

Let $\underline{A}$ be a given vector defined with respect to two general curvilinear coordinate systems (u_1,u_2,u_3) and (v_1,v_2,v_3). Suppose that the transformation equations from a rectangular (x,y,z) system to each of the two systems are given by

$$\begin{cases} x = x_1(u_1,u_2,u_3) \quad , y = y_1(u_1,u_2,u_3) \; , z = z_1(u_1,u_2,u_3) \\ x = x_2(v_1,v_2,v_3) \quad , y = y_2(v_1,v_2,v_3) \; , \; z = z_2(v_1,v_2,v_3) \end{cases} \qquad (i)$$

Then there exists a transformation directly from the (u_1,u_2,u_3) to the (v_1,v_2,v_3) defined by

$$u_1 = u_1(v_1,v_2,v_3) \; , \; u_2 = u_2(v_1,v_2,v_3) \quad , u_3 = u_3(v_1,v_2,v_3) \qquad (ii)$$

and conversely. From (i)

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3 = \underline{\alpha}_1 du_1 + \underline{\alpha}_2 du_2 + \underline{\alpha}_3 du_3 \quad \ldots(5.24)$$

$$d\underline{r} = \frac{\partial \underline{r}}{\partial v_1} dv_1 + \frac{\partial \underline{r}}{\partial v_2} dv_2 + \frac{\partial \underline{r}}{\partial v_3} dv_3 = \underline{\beta}_1 dv_1 + \underline{\beta}_2 dv_2 + \underline{\beta}_3 dv_3$$

Then

$$\underline{\alpha}_1 du_1 + \underline{\alpha}_2 du_2 + \underline{\alpha}_3 du_3 = \underline{\beta}_1 dv_1 + \underline{\beta}_2 dv_2 + \underline{\beta}_3 dv_3 \qquad (iii)$$

From(ii)

$$\begin{cases} du_1 = \frac{\partial u_1}{\partial v_1} dv_1 + \frac{\partial u_1}{\partial v_2} dv_2 + \frac{\partial u_1}{\partial v_3} dv_3 \\ du_2 = \frac{\partial u_2}{\partial v_1} dv_1 + \frac{\partial u_2}{\partial v_2} dv_2 + \frac{\partial u_2}{\partial v_3} dv_3 \\ du_3 = \frac{\partial u_3}{\partial v_1} dv_1 + \frac{\partial u_3}{\partial v_2} dv_2 + \frac{\partial u_3}{\partial v_3} dv_3 \end{cases} \qquad \ldots(5.25)$$

Substituting into (iii) and equating coefficients of dv_1, dv_2, dv_3 on both sides, we find

$$\begin{aligned} \underline{\beta}_1 &= \underline{\alpha}_1 \frac{\partial u_1}{\partial v_1} + \underline{\alpha}_2 \frac{\partial u_2}{\partial v_1} + \underline{\alpha}_3 \frac{\partial u_3}{\partial v_1} \\ \underline{\beta}_2 &= \underline{\alpha}_1 \frac{\partial u_1}{\partial v_2} + \underline{\alpha}_2 \frac{\partial u_2}{\partial v_2} + \underline{\alpha}_3 \frac{\partial u_3}{\partial v_2} \\ \underline{\beta}_3 &= \underline{\alpha}_1 \frac{\partial u_1}{\partial v_3} + \underline{\alpha}_2 \frac{\partial u_2}{\partial v_3} + \underline{\alpha}_3 \frac{\partial u_3}{\partial v_3} \end{aligned} \qquad (iv)$$

a different form

opposite against

A Contravariant Vector:

Now $\underline{A}$ can be expressed in the two coordinate systems as

$$\underline{A} = C_1 \underline{\alpha}_1 + C_2 \underline{\alpha}_2 + C_3 \underline{\alpha}_3 \quad \& \quad \underline{A} = \overline{C}_1 \underline{\beta}_1 + \overline{C}_2 \underline{\beta}_2 + \overline{C}_3 \underline{\beta}_3 \quad \ldots(5.26)$$

where C_1, C_2, C_3 and $\overline{C}_1, \overline{C}_2, \overline{C}_3$ are the contravariant components of $\underline{A}$.,

$$\underline{F} = (x, y, z) = \big(x(u_1, u_2, u_3),\ y(u_1, u_2, u_3),\ z(u_1, u_2, u_3)\big) \quad ①$$

$$\underline{F} = (u_1, u_2, u_3) = \big(u_1(x, y, z),\ u_2(x, y, z),\ u_3(x, y, z)\big) \quad ②$$

but $\exists\ \underline{F} = (x, y, z) = \big(x(v_1, v_2, v_3),\ y(v_1, v_2, v_3),\ z(v_1, v_2, v_3)\big) \quad ③$

and also $\underline{F} = (v_1, v_2, v_3) = \big(v_1(x, y, z),\ v_2(x, y, z),\ v_3(x, y, z)\big) \quad ④$

from ② & ③ $\Rightarrow$

$$\underline{F} = (u_1, u_2, u_3) = \Big(u_1\big(x(v_1, v_2, v_3),\ y(v_1, v_2, v_3),\ z(v_1, v_2, v_3)\big)$$

$$u_2\big(\quad " \quad " \quad\big)$$

$$u_3\big(\quad " \quad\big)\Big)$$

$$= u_1(v_1, v_2, v_3),\ u_2(v_1, v_2, v_3),\ u_3(v_1, v_2, v_3)$$

Substituting (iv) into (5.26)

$$C_1\underline{\alpha}_1+C_2\underline{\alpha}_2+C_3\underline{\alpha}_3=\left(\bar{C}_1\frac{\partial u_1}{\partial v_1}+\bar{C}_2\frac{\partial u_1}{\partial v_2}+\bar{C}_3\frac{\partial u_1}{\partial v_3}\right)\underline{\alpha}_1$$
$$+\left(\bar{C}_1\frac{\partial u_2}{\partial v_1}+\bar{C}_2\frac{\partial u_2}{\partial v_2}+\bar{C}_3\frac{\partial u_2}{\partial v_3}\right)\underline{\alpha}_2$$
$$+\left(\bar{C}_1\frac{\partial u_3}{\partial v_1}+\bar{C}_2\frac{\partial u_3}{\partial v_2}+\bar{C}_3\frac{\partial u_3}{\partial v_3}\right)\underline{\alpha}_3$$

Then

$$\left\{\begin{aligned} C_1&=\bar{C}_1\frac{\partial u_1}{\partial v_1}+\bar{C}_2\frac{\partial u_1}{\partial v_2}+\bar{C}_3\frac{\partial u_1}{\partial v_3} \\ C_2&=\bar{C}_1\frac{\partial u_2}{\partial v_1}+\bar{C}_2\frac{\partial u_2}{\partial v_2}+\bar{C}_3\frac{\partial u_2}{\partial v_3} \\ C_3&=\bar{C}_1\frac{\partial u_3}{\partial v_1}+\bar{C}_2\frac{\partial u_3}{\partial v_2}+\bar{C}_3\frac{\partial u_3}{\partial v_3}\end{aligned}\right. \qquad \ldots(5.27)$$

i.e.

$$C_p=\bar{C}_1\frac{\partial u_p}{\partial v_1}+\bar{C}_2\frac{\partial u_p}{\partial s_2}+\bar{C}_3\frac{\partial u_p}{\partial v_3}\quad ;\ p=1,2,3 \qquad \ldots(5.28)$$

or

$$C_p=\sum_{q=1}^{3}\bar{C}_q\frac{\partial u_p}{\partial v_q}\quad ;\ p=1,2,3 \qquad \ldots(5.29)$$

Similarly, by interchanging the ccoordinates we see that

$$\bar{C}_p=\sum_{q=1}^{3}C_q\frac{\partial v_p}{\partial u_q}\quad ;\ p=1,2,3 \qquad \ldots(5.30)$$

If three quantities C_1, C_2, C_3 of a coordinate system (u_1, u_2, u_3) are related to three other quantities $\bar{C}_1, \bar{C}_2, \bar{C}_3$ of another coordinate system (v_1, v_2, v_3) by the transformation equations, (5.27), (5.28), (5.29) or (5.30), then the quantities are called components of a contravariant vector or a contravariant tensor of the first rank.

A Covariant Vector:

If we express $\underline{A}$ in the form

$$\underline{A} = c_1 \nabla u_1 + c_2 \nabla u_2 + c_3 \nabla u_3 = \bar{c}_1 \nabla v_1 + \bar{c}_2 \nabla v_2 + \bar{c}_3 \nabla v_3 \qquad \ldots(5.31)$$

Now since $v_p = v_p(u_1, u_2, u_3)$ with $p=1,2,3$.

$$\begin{aligned}
\frac{\partial v_p}{\partial x} &= \frac{\partial v_p}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial v_p}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial v_p}{\partial u_3}\frac{\partial u_3}{\partial x} \\
\frac{\partial v_p}{\partial y} &= \frac{\partial v_p}{\partial u_1}\frac{\partial u_1}{\partial y} + \frac{\partial v_p}{\partial u_2}\frac{\partial u_2}{\partial y} + \frac{\partial v_p}{\partial u_3}\frac{\partial u_3}{\partial x} \qquad \ldots(5.32) \\
\frac{\partial v_p}{\partial z} &= \frac{\partial v_p}{\partial u_1}\frac{\partial u_1}{\partial z} + \frac{\partial v_p}{\partial u_2}\frac{\partial u_2}{\partial z} + \frac{\partial v_p}{\partial u_3}\frac{\partial u_3}{\partial z} \quad ; \; p = 1,2,3
\end{aligned}$$

Also,

$$\begin{aligned}
c_1 \nabla u_1 + c_2 \nabla u_2 + c_3 \nabla u_3 = &\left(c_1 \frac{\partial u_1}{\partial x} + c_2 \frac{\partial u_2}{\partial x} + c_3 \frac{\partial u_3}{\partial x}\right)\underline{i} \\
&+ \left(c_1 \frac{\partial u_1}{\partial y} + c_2 \frac{\partial u_2}{\partial y} + c_3 \frac{\partial u_3}{\partial y}\right)\underline{j} \\
&+ \left(c_1 \frac{\partial u_1}{\partial z} + c_2 \frac{\partial u_2}{\partial z} + c_3 \frac{\partial u_3}{\partial z}\right)\underline{k}
\end{aligned}$$

and

$$\begin{aligned}
\bar{c}_1 \nabla v_1 + \bar{c}_2 \nabla v_2 + \bar{c}_3 \nabla v_3 = &\left(\bar{c}_1 \frac{\partial v_1}{\partial x} + \bar{c}_2 \frac{\partial v_2}{\partial x} + \bar{c}_3 \frac{\partial v_3}{\partial x}\right)\underline{i} + \left(\bar{c}_1 \frac{\partial v_1}{\partial y} + \bar{c}_2 \frac{\partial v_2}{\partial y} + \bar{c}_3 \frac{\partial v_3}{\partial y}\right)\underline{j} \\
&+ \left(\bar{c}_1 \frac{\partial v_1}{\partial z} + \bar{c}_2 \frac{\partial v_2}{\partial z} + \bar{c}_3 \frac{\partial v_3}{\partial z}\right)\underline{k}
\end{aligned}$$

Equating coefficients of $\underline{i}, \underline{j}, \underline{k}$,

$$\left\{\begin{aligned}
c_1 \frac{\partial u_1}{\partial x} + c_2 \frac{\partial u_2}{\partial x} + c_3 \frac{\partial u_3}{\partial x} &= \bar{c}_1 \frac{\partial v_1}{\partial x} + \bar{c}_2 \frac{\partial v_2}{\partial x} + \bar{c}_3 \frac{\partial v_3}{\partial x} \\
c_1 \frac{\partial u_1}{\partial y} + c_2 \frac{\partial u_2}{\partial y} + c_3 \frac{\partial u_3}{\partial y} &= \bar{c}_1 \frac{\partial v_1}{\partial y} + \bar{c}_2 \frac{\partial v_2}{\partial y} + \bar{c}_3 \frac{\partial v_3}{\partial y} \\
c_1 \frac{\partial u_1}{\partial z} + c_2 \frac{\partial u_2}{\partial z} + c_3 \frac{\partial u_3}{\partial z} &= \bar{c}_1 \frac{\partial v_1}{\partial z} + \bar{c}_2 \frac{\partial v_2}{\partial z} + \bar{c}_3 \frac{\partial v_3}{\partial z}
\end{aligned}\right.$$

Substituting equations(5.32) with $p=1,2,3$ in any of the above eqs. and equating coefficients of $\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, \frac{\partial u_3}{\partial x}, \frac{\partial u_1}{\partial y}, \frac{\partial u_2}{\partial y}, \frac{\partial u_3}{\partial y}, \frac{\partial u_1}{\partial z}, \frac{\partial u_2}{\partial z}, \frac{\partial u_3}{\partial z}$

we find

$$\begin{cases} c_1 = \bar{c}_1 \dfrac{\partial v_1}{\partial u_1} + \bar{c}_2 \dfrac{\partial v_2}{\partial u_1} + \bar{c}_3 \dfrac{\partial v_3}{\partial u_1} \\ c_2 = \bar{c}_1 \dfrac{\partial v_1}{\partial u_2} + \bar{c}_2 \dfrac{\partial v_2}{\partial u_2} + \bar{c}_3 \dfrac{\partial v_3}{\partial u_2} \\ c_3 = \bar{c}_1 \dfrac{\partial v_1}{\partial u_3} + \bar{c}_2 \dfrac{\partial v_2}{\partial u_3} + \bar{c}_3 \dfrac{\partial v_3}{\partial u_3} \end{cases} \qquad \ldots(5.33)$$

which can be written

$$c_p = \bar{c}_1 \frac{\partial v_1}{\partial u_p} + \bar{c}_2 \frac{\partial v_2}{\partial u_p} + \bar{c}_3 \frac{\partial v_3}{\partial u_p} \qquad p = 1,2,3$$

or

$$c_p = \sum_{q=1}^{3} \bar{c}_q \frac{\partial v_q}{\partial u_p} \quad ; \; p = 1,2,3 \qquad \ldots(5.34)$$

Similarly, we ca show that

$$\bar{c}_p = \sum_{q=1}^{3} c_q \frac{\partial u_q}{\partial v_p} \quad ; \; p = 1,2,3 \qquad \ldots(5.35)$$

If three quantities c_1, c_2, c_3 of a coordinate system (u_1, u_2, u_3) are related to three other quantities $\bar{c}_1, \bar{c}_2, \bar{c}_3$ of another coordinate system (v_1, v_2, v_3) by the transformation equations (5.33), (5.34) or (5.35), then the quantities are called components of a covariant vector or a covariant tensor of the first rank.

On the other hand, a vector field $\underline{A}$ is called an invariant if $\underline{A}(u_1, u_2, u_3) = \underline{A}(v_1, v_2, v_3)$ under the transformation between coordinates of the two systems (u_1, u_2, u_3) and (v_1, v_2, v_3).

In generalizing the concepts in the above section to higher-dimensional spaces, and in generalizing the concept of vector, we are led to tensor analysis.

5.7 Tensor Fields:

In three dimensional space, a point is a set of three numbers, e.g. (x,y,z), (ρ,θ,z), (r,θ,ϕ). A point in N dimensional space is, by analogy, aset of N numbers denoted by $(x^1,x^2,\ldots\ldots,x^N)$. Let $(x^1,x^2,\ldots,x^N)$ and $(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^N)$ be coordinates of a point in two different frames of reference. The equations of transformation from one framo to another are given by

$$x^k = x^k(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^N)$$
$$\bar{x}^k = \bar{x}^k(x^1,x^2,\ldots,x^N) \qquad \ldots(5.36)$$

or

If N quantities $A^1,A^2,\ldots,A^N$ in a coordinate system $(x^1,x^2,\ldots,x^N)$ are related to N other quantities $\bar{A}^1,\bar{A}^2,\ldots,\bar{A}^N$ in another coordinate system $(\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^N)$ by the transformation equations

$$\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} A^q \quad , p=1,2,\ldots,N$$

which by the summation convention can simply be written as

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q \qquad \ldots(5.37)$$

they are called components of a contravariant vector or contravariant tensor of the first rank or first order.

Similarly

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q \qquad \ldots(5.38)$$

are called components of a covariant vector or covariant tensor of the first rank or first order.

If to each point of a region in N dimensional space there corresponds a definite tensor, we say that a tensor field has been defined. This is a vector field or a scalar field according as the tensor is of rank one or zero.

5.8 The Line Element & Metric Tensor:

In rectangular coordinates (x,y,z) the differential of arc-length ds is defined as

$$ds^2 = dx^2 + dy^2 + dz^2$$

By transforming to general curvilinear coordinates we have

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u_1} du_1 + \frac{\partial \underline{r}}{\partial u_2} du_2 + \frac{\partial \underline{r}}{\partial u_3} du_3 = \underline{\alpha}_1 du_1 + \underline{\alpha}_2 du_2 + \underline{\alpha}_3 du_3$$

Then

$$\begin{aligned} ds^2 = d\underline{r}\cdot d\underline{r} = \underline{\alpha}_1\cdot\underline{\alpha}_1 du_1^2 + \underline{\alpha}_1\cdot\underline{\alpha}_2 du_1 du_2 + \underline{\alpha}_1\cdot\underline{\alpha}_3 du_1 du_3 \\ + \underline{\alpha}_2\cdot\underline{\alpha}_1 du_2 du_1 + \underline{\alpha}_2\cdot\underline{\alpha}_2 du_2^2 + \underline{\alpha}_2\cdot\underline{\alpha}_3 du_2 du_3 \\ + \underline{\alpha}_3\cdot\underline{\alpha}_1 du_3 du_1 + \underline{\alpha}_3\cdot\underline{\alpha}_2 du_3 du_2 + \underline{\alpha}_3\cdot\underline{\alpha}_3 du_3^2 \end{aligned}$$

or

$$ds^2 = \sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq}\, du_p\, du_q \qquad \ldots(5.39)$$

Such spaces are called three dimensional Euclidean spaces and Eq.(5.39) is called the fundamental quadratic form or metric form. The quantities g_{pq} are called metric coefficients and are symmetric,i.e. $g_{pq}=g_{qp}$ .If $g_{pq}=0$, $p\neq q$,then the coordinate system is orthogonal. In this case $g_{11}=h_1^2$,$g_{22}=h_2^2$,$g_{33}=h_3^2$.

In N dimensional space,the metric form is written as

$$ds^2 = g_{pq}\, dx^p\, dx^q \qquad \ldots(5.40)$$

using the summation convention.

In the special case where there exists a transformation of coordinates from x^j to $\bar{x}^k$ such that the metric form is transformed into

$$(d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \text{----} \text{--} + (d\bar{x}^N)^2 \qquad \text{or} \quad d\bar{x}^k\, d\bar{x}^k$$

then the space is called N dimensional Euclidean space.

In the general case, however, the space is called Riemannian. The quantities g_{pq} are the components of a covariant tensor of rank two called the metric tensor or fundamental tensor. In matrix form the metric tensor can be written
$$\begin{pmatrix} g_{11} & g_{12} & --- & g_{1N} \\ g_{21} & g_{22} & --- & g_{2N} \\ -- & -- & --- & -- \\ g_{N1} & g_{N2} & --- & g_{NN} \end{pmatrix}$$
and g is its determinant.

5.9 Geodesics:

The distance s between two points t_1 & t_2 on a curve $x^r = x^r(t)$ in a Riemannian space is given by

$$s = \int_{t_1}^{t_2} \sqrt{g_{pq} \frac{dx^p}{dt} \cdot \frac{dx^q}{dt}}\, dt \qquad \ldots(5.41)$$

That curve in the space which makes the distance a minimum is called a geodesic of the space.

By the use of the calculus of variation, the geodesics are found from the differential equation

$$\frac{d^2x^r}{ds^2} + \left\{ {r \atop pq} \right\} \frac{dx^p}{ds} \cdot \frac{dx^q}{ds} = 0 \qquad \ldots(5.42)$$

where s is the arc length.

The symbols
$$[pq,r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{pr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right)$$
$$\left\{ {s \atop pq} \right\} = g^{sr}[pq,r] \qquad \ldots(5.43)$$

are called the Christoffel symbols of the first and second kind respectively.

5.10 Tensor Form of Gradient, Divergence & Curl:

The covariant derivative of a tensor A_p with respect to x^q is denoted by

$$A_{p,q} \equiv \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s \qquad \ldots(5.43)$$

a covariant tensor of rank two.

The covariant derivative of a tensor A^p with respect to x^q is denoted by $A^p_{,q}$

$$A^p_{,q} \equiv \frac{\partial A^p}{\partial x^q} + \left\{ {q \atop ps} \right\} A^s \qquad \ldots(5.44)$$

a mixed tensor of rank two.

For rectangular systems, the Christoffel symbols are zero and the covariant derivatives are the usual partial derivatives.

If ϕ is a scalar or invariant, the gradient of ϕ is defined by

$$\nabla\varphi = \text{grad}\,\varphi = \varphi_{,p} = \frac{\partial \varphi}{\partial x^p} \qquad \ldots(5.45)$$

where $\phi_{,p}$ is the covariant derivative of ϕ with respect to x^p.

The divergence of A^p is defined by

$$A^p_{,p} = \text{div}\, A^p = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g}\, A^k \right) \qquad \ldots(5.46)$$

The curl of A_p is

$$A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p} \qquad \ldots(5.47)$$

a tensor of rank two.

The Laplacian of ϕ is

$$\nabla^2\varphi = \text{div}\,\varphi_{,p} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g}\, g^{ik} \frac{\partial \varphi}{\partial x^k} \right) \qquad \ldots(5.48)$$

Problems-V-Curvilinear Coordinates

5.1 :Represent the vector $\underline{A} = z\underline{i} - 2x\underline{j} + y\underline{k}$ in cylindrical coordinates. Then determine A_ρ, A_θ, A_z.

5.2 :The transformation from rectangular coordinates to parabolic cylindrical coordinates is defined by the equations

$$x = \frac{1}{2}(u^2 - v^2) \quad , y = uv \quad , z = z$$

Prove that the system is orthogonal. Find dr^2 and the scale factors. Find the Jacobian of the transformation and the volume elements.

5.3 :Write $\operatorname{div} \underline{A}$ and $\nabla^2 \varphi$ in parabolic cylindrical coordinates.

5.4 :Prove that the acceleration of a moving particle in spherical coordinates, is given by

$$(\ddot{r} - r\dot{\theta}^2 - r\dot{\psi}^2 \sin^2\theta)\underline{e}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\psi}^2 \sin\theta\cos\theta)\underline{e}_2$$
$$+ (2\dot{r}\dot{\psi}\sin\theta + 2r\dot{\theta}\dot{\psi}\cos\theta + r\ddot{\psi}\sin\theta)\underline{e}_3$$

where dots denote time derivatives and $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ are unit-vectors in the directions of increasing (r, θ, ψ).

5.5 :If $F = x + 3y^2 - z^3$, $G = 2x^2yz$, $H = 2x^3 - xy$, evaluate $\frac{\partial(F, G, H)}{\partial(x, y, z)}$ at $(1, -1, 0)$.

5.6 :If $F = xy + yz + zx$, $G = x^2 + y^2 + z^2$ and $H = x + y + z$, determine whether there is a functional relationship connecting F, G, H, if so find it.

5.7 : If $F(P,V,T) = 0$, prove that

$$\left.\frac{\partial P}{\partial T}\right|_V \left.\frac{\partial T}{\partial V}\right|_P = -\left.\frac{\partial P}{\partial V}\right|_T \quad \& \quad \left.\frac{\partial P}{\partial T}\right|_V \left.\frac{\partial T}{\partial V}\right|_P \left.\frac{\partial V}{\partial P}\right|_T = -1$$

where a subscript indicates the varibale which is to be held constant. (The results are useful in thermodynamics where P, V, T correspond to pressure, volume, and temperature of a physical system).

5.8 : If $x = \xi(u,v,w)$, $y = \eta(u,v,w)$ & $z = \zeta(u,v,w)$

prove that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$$

provided $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

Give an interpretation of the result in terms of transformation.

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VI- VECTOR FIELDS

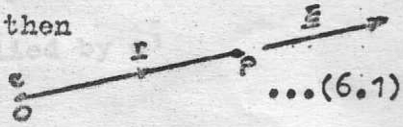
A. ELECTROSTATIC FIELD

6.1 The Electrostatic Potential:

If a unit charge is placed at some point p in the presence of given fixed charges, it will in general experience a force. This force is defined as the electrostatic field, or simply the electric field, at p. Since it has a magnitude as well as direction the field is a vector quantity, denoted by $\underline{E}$.

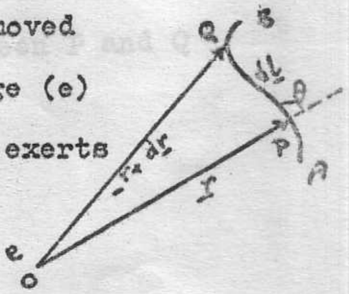
Let us calculate the electric field due to a single point charge (e) at a point (o), $\underline{op} = \bar{\underline{r}}$. A unit charge at p will experience a force e/r^2 along the line $\underline{op}$. The field at p is therefore of magnitude e/r^2 and direction $\overrightarrow{op}$. If $\underline{r}$ denotes the vector $\underline{op}$, then

$$\underline{E} = e\underline{r}/r^3 \qquad \ldots(6.1)$$



Let us suppose that the unit test charge is moved from A to B under the influence of the fixed charge (e) at o. In going a distance $d\underline{l}$ from p to Q the filed exerts a forwe $\underline{E}$ and therefore does work

$$\underline{E}.d\underline{l} = \frac{e}{r^2}\cos\theta\, dl = \frac{e\,dr}{r^2}$$



where θ is the angle between $\underline{op}$ and $\underline{oQ}$. The total work done by the field in moving the charge from A to B is therefore

$$\int_A^B \frac{e\,dr}{r^2} = \left(\frac{e}{r_A} - \frac{e}{r_B}\right)$$

Work done against the field is

$$\int_A^B \underline{E}.d\underline{l} = \frac{e}{r_B} - \frac{e}{r_A} \qquad \ldots(6.2)$$

This depends only on the position of A and B and not on the path connecting them. Thus the field is conservative.

The quantity $(e/r_B-e/r_A)$ is called the difference of potential between A and B and is written V_B-V_A. Thus, the difference of potential between two points represents the work to be done in order to move a unit test charge from one point to the other. We generally define the absolute value of the potential at a point by the condition that if A is at infinity $V_A=0$. V_B then represents the work to be done in bringing unit charge from infinity to B, $V_B=e/r_B$. In other words, the potential due to a charge e, at a point of position vector $\underline{\bar{r}}$, is

$$V = e/r \qquad \ldots(6.3)$$

If our test charge has magnitude (e'), instead of unity, the law of force shows that the force on it is merely multiplied by e'; therefore the work done from A to B is also multiplied by e'

$$e'(V_B-V_A)$$

Consider the difference of potential dV between P and Q

$dV=V_Q-V_P$=work done on unit charge from P to Q

= - work done by the field.

$= -\underline{\bar{E}}.d\underline{\bar{l}}$

$$\therefore dV = -(E_1 dx + E_2 dy + E_3 dz)$$

Therefore

$$E_1 = -\frac{\partial V}{\partial x}, \quad E_2 = -\frac{\partial V}{\partial y}, \quad E_3 = -\frac{\partial V}{\partial z}$$

or

$$\left.\begin{aligned} \underline{E} &= -\nabla V \\ \text{i.e.} \quad V_B-V_A &= -\int_A^B \underline{E}.d\underline{l} \end{aligned}\right\} \qquad \ldots(6.4)$$

6.2 Field due to Several Charges:

If instead of one charge e we have two distinct charges, then their effects on any other charge are simply additive. Thus two point charges e_1 and e_2 exert on a third carge e a force equal to the sum of the two forces $e.e_1/r_1^2$ and $e.e_2/r_2^2$, added vectorially in the usual manner for forces. This constitutes the principle of superposition, which states that for any given system of charges, the final force and potential are the same as if we simply superposed the separate forces and potentials.

$$V = V_1 + V_2 + \cdots = \sum \frac{e_i}{r_i}$$

&

$$\underline{E} = \underline{E}_1 + \underline{E}_2 + \cdots = \sum \frac{e_i \underline{r}_i}{r_i^3} \qquad \ldots(6.5)$$

$\underline{r}_i$ is, of course, the vector from one of the given charges to the point at which $\underline{E}$ is required.

It sometimes happens that our fixed charges are not all separate and distinct, but that they are distributed through a given volume or on a given surface. We speak of a volume density ρ, if the charge in volume $d\tau$, is $\rho d\tau$; and of surface density σ, if the charge on an element of surface dS is σ dS. Thus

$$V = \iiint \frac{\rho d\tau}{r} + \iint \frac{\sigma dS}{r} \qquad \ldots(6.6)$$

6.3 Equipotentials and Lines of Force:

The equipotential surface is that surface on which every point is at the same potential, its equation is therefore,

$$V = \text{const.}$$

The line of force is that line on which, at any point, the direction of $\underline{E}$ coincides with the tangent to the line at this point.

If we take two points P &Q on an equipotential surface, $V_p = V_Q$ and hence the work done on unit positive charge by electric force is

$$\underline{E} \cdot \delta \underline{r} = 0$$

Since $|\delta \underline{r}| \neq 0$, we have either $|\underline{E}| = 0$ or $\underline{E}$ is perpendicular to $\delta \underline{r}$. Therefore, the direction of $\underline{E}$ at any point is normal to the equipotential surface through that point. In other words, the lines of force cut equipotential surfaces at right angles, except at points where the intensity vanishes (Points of equilibrium). This property is also clear from the r relation

$$\underline{E} = -\nabla V$$

Since the direction of the lines of force is the same as the direction of $\underline{E}$, the lines of force must be defined by

$$\frac{dx}{E_1} = \frac{dy}{E_2} = \frac{dz}{E_3}$$

in cartesian coordinates, and

$$\frac{h_1\,du}{E_u} = \frac{h_2\,dv}{E_v} = \frac{h_3\,dw}{E_w} \qquad \ldots(6.7)$$

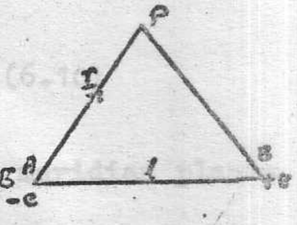
in orthogonal curvilinear coordinates.

It is clear that neither the equipotentials nor the lines of force intersect, otherwise the V and $\underline{E}$ will not be a single-valued function at the curves or points of intersection, respectively.

In the particular case of a single charge at a point, the lines of force are radii, drawn from the point in all directions. There is one and only one radius through any point, and all these lines cut the equipotentials, which are concentric spheres around the point, at right angles.

6.4 The Electric Dipole:

It frequently happens that we have a pair of equal and opposite charges $\pm e$ at a distance l apart, e is usually very large and l is very small in such a way that the product el has a finite value m. Such a combination is called an electric dipole.



We define the moment of a dipole by the quantity of magnitude m and in the direction from the negative to the positive charge. If we have a dipole $\underline{m}$ at $\underline{AB}$, the resulting potential at p is

$$V = -e/AP + e/BP$$

$$= E \times \text{change in } (1/r) \text{ from A to B}$$

For points at distances r large compared with l, we get

$$V \simeq el \frac{\partial}{\partial s}(1/r)$$

where $\frac{\partial}{\partial s}$ denotes differentiation in the direction AB.

If $\underline{r}$ is the position vector of p, with respect to A, i.e. $\underline{r} = \underline{AP}$, then

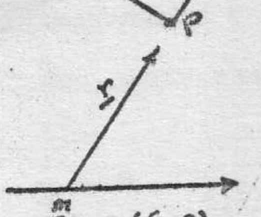
$$V_p = \underline{m}.\text{grad}_A(1/r)$$

$$V = \underline{m}.\text{grad}_p(1/r)$$

i.e.
$$V_p = (\underline{m}.\underline{r})/r^3 = m\cos\theta/r^2 \qquad \ldots(6.8)$$

The electric field at p is most conveniently expressed in terms of the radial and transverse components E_r and E_θ.

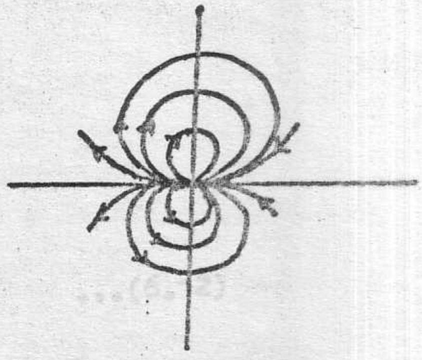
Since



$$\underline{E} = -\text{grad } V$$

$$\left.\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = \frac{2m\cos\theta}{r^3} \\ E_\theta &= -\frac{\partial V}{r\partial\theta} = \frac{m\sin\theta}{r^3} \\ E_\psi &= -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\psi} = 0 \end{aligned}\right\} \qquad \ldots(6.9)$$

The equation of lines of force is



$$\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta} = \frac{r\sin\theta\,d\phi}{E_\phi}$$

or $$\frac{dr}{r} = 2\cot\theta\,d\theta \quad \text{and } d\phi = 0$$

Thus
$$\ln r = 2\ln(\sin\theta) + \ln C$$

or $$r = C\sin^2\theta \qquad \& \ \phi = \text{Const.} = C' \qquad \ldots(6.10)$$

i.e. the lines of force are the curves $r = c\sin^2\theta$ in the meridian planes $\phi = c'$.

The electric field at p due to a dipole $\underline{\bar{m}}$ is

$$\underline{E} = -\operatorname{grad}\left(\frac{\underline{m}\cdot\underline{r}}{r^3}\right)$$

$$= -\frac{1}{r^3}\operatorname{grad}(\underline{m}\cdot\underline{r}) + (\underline{m}\cdot\underline{r})\operatorname{grad}(1/r^3)$$

$$\therefore\ \underline{E} = -\frac{\underline{m}}{r^3} + \frac{3(\underline{m}\cdot\underline{r})\,\underline{r}}{r^5} \qquad \ldots(6.11)$$

6.5 Flux and Gauss Theorem:

If S is any surface bounded by a closed contour C, the flux of the electric field $\underline{E}$ across the surface is defined by the surface integral

$$N = \iint_S \underline{E}\cdot d\underline{S}$$

If the field is due to a point charge e at the origin O, then

$$\underline{E} = \frac{e}{r^3}\,\underline{r}$$

$$\therefore\ N = \iint_S \underline{E}\cdot d\underline{S} = e\iint_S \frac{\underline{r}\cdot d\underline{S}}{r^3} = e\,\varpi$$

where ϖ is the solid angãe subtended by the surface at O.

If S is a closed surface, then

$$N = \begin{cases} 4\pi e & \text{, if } e \text{ is inside } S, \\ 2\pi e & \text{, if } e \text{ is on } S \\ 0 & \text{, if } e \text{ is outside } S \end{cases} \qquad \dots(6.12)$$

If there are several point charges $e_1, e_2, \dots, e_n$ within S, the total charge is $Q = \sum e_i$ and

$$N = 4\pi Q \qquad \dots(6.13)$$

This is known as Gauss theorem.

If we also have several point charges $e'_1, e'_2, \dots, e'_n$ on the surface S, then $Q' = \sum e'_i$ is the total charge residing on the surface. The flux is

$$N = 4\pi Q + 2\pi Q' \qquad \dots(6.14)$$

Suppose that the electric field is due to a continuous distribution of charge of density ρ per unit volume, within a closed surface S. Then if $d\tau$ is the volume wlement, using Gauss theorem we have

$$\iint_S \underline{E} \cdot d\underline{s} = 4\pi \iiint_\tau \rho \, d\tau$$

But by Gauss divergence theorem

$$\iint_S \underline{E} \cdot d\underline{s} = \iiint_\tau \operatorname{div} \underline{E} \, d\tau$$

so $$(\operatorname{div} E - 4\pi\rho)\, d\tau = 0$$

The equation holds for any volume, and since the integrand is continuous, this implies that $(\operatorname{div} E - 4\pi\rho)$ itself must be zero. Gauss theorem may therefore be written in the form

$$\operatorname{div} \underline{E} = 4\pi\rho \qquad \dots(6.15)$$

Since $\operatorname{div} E = -\operatorname{div} \operatorname{grad} V = -\nabla^2 V$

Then

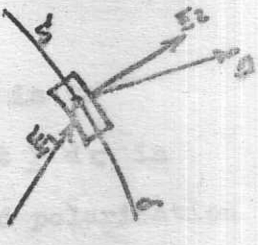
$$\nabla^2 V = -4\pi\rho \qquad \text{Poisson's Equation} \ldots (6.16)$$

In the particular case of zero charge density, $\rho = 0$, it becomes

$$\nabla^2 V = 0 \qquad \text{Laplace's Equation} \ldots (6.17)$$

6.6 Gauss Theorem for Surface Distribution:

Let S be a surface on which there is a charge distribution σ per unit area. Let $\underline{E}_1$ and $\underline{E}_2$ be the electric fields at a point p on the two sides of S. On the element of area δS at p erect a small cylinder whose generators are normal to δS and so small compared with the linear dimensions of δS, that the flux of E across the curved sides of the cylinder can be neglected.



Applying Gauss theorem we get

$$(\underline{E}_2 - \underline{E}_1)\cdot \underline{n}\, \delta S = 4\pi\sigma\, \delta S$$

Divide by δS and let the element shrink to p we get

$$\underline{E}_2 \cdot \underline{n} - \underline{E}_1 \cdot \underline{n} = 4\pi\sigma$$

or

$$\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} = -4\pi\sigma \qquad \ldots (6.18)$$

This shows that there is a discontinuity across the surface of value $4\pi\sigma$, in the normal component of the field E. This is Gauss theorem for surface distribution.

6.7 Dielectrics and Polarization Vector:

If the space between the two plates of a condenser is filled with an insulating material, such as glass or mica, its capacity will be multiplied by a certain constant K, which depends on the material but is independent of the shape of the condenser. Such substances are called dielectrics .

and K is the dielectric constant; K=1 for free space.

In the absence of the electric field, the molecules of the dielectric (which are dipoles) will point equally often in all directions. However, when the field is applied, there will be a couple tending to orient them relative to the field. If there are N molecules in unit volume, the total moment induced by an electric field E is

$$\underline{P} = N\alpha\,\underline{E} \quad \text{per unit volume} \qquad \ldots(6.19)$$

$\underline{P}$ is the polarization vector and α is the polarizability.

Thus the effect of an electric field $\underline{E}$ upon a dielectric is to create polarization $\underline{P}$, with which there is an associated moment $\underline{P}\,d\tau$ in each volume element $d\tau$. The potential arising from the induced polarization is

$$V = \iiint_{\tau} \underline{P}\cdot \operatorname{grad}(1/r)\,d\tau$$

But

$$\operatorname{div}\frac{\underline{P}}{r} = \frac{1}{r}\operatorname{div}\underline{P} + \underline{P}\cdot\operatorname{grad}(1/r)$$

Then

$$V = \iiint_{\tau} \operatorname{div}\frac{\underline{P}}{r}\,d\tau - \iiint_{\tau}\frac{1}{r}\operatorname{div}\underline{P}\,d\tau$$

$$V = \iint_{S}\frac{\underline{P}\cdot d\underline{S}}{r} - \iiint_{\tau}\frac{1}{r}\operatorname{div}\underline{P}\,d\tau \qquad \ldots(6.20)$$

i.e. the resulting potential is exactly the same as we should obtain if we were to suppose a volume distribution of charge $-\operatorname{div}\underline{P}$, throughout the dielectric and a surface distribution P_n on the boundary of the dielectric.

Suppose that at any point in the medium the field is $\underline{E}$, with resulting polarization $\underline{P}$. In addition to any true charges ρ and σ that may be present, we must include contributions to the potential from the apparent charges $-\operatorname{div}\underline{P}$ and P_n which take account of the polarization.

Gauss'flux theorem,therefore,takes the form

$$\operatorname{div} \underline{E} = 4\pi(\rho - \operatorname{div} \underline{P})$$

i.e. $$\operatorname{div}(\underline{E} + 4\pi \underline{P}) = 4\pi\rho$$

or $$\operatorname{div} \underline{D} = 4\rho \qquad \ldots(6.21)$$

where $$\underline{D} = \underline{E} + 4\pi \underline{P} \qquad \ldots(6.22)$$

$\underline{D}$ is called the electric displacement vector.In isotropic material,

$$\underline{D} = \underline{E} + 4\pi \underline{P} = (1+4\pi k)\underline{E} = K\underline{E} \qquad \ldots(6.23)$$

K is the dielectric constant.

6.8 Energy of the Electrostatic Field:

Let $r_{12}, r_{13}, \ldots$ denote the distances between charges e_1 and e_2, e_1 and e_3, etc.Then the mutual potential energy is clearly $\sum\sum_{i\,j} e_i e_j / r_{ij}$ where the double summation includes all pairs of distinct values of i and j. We may write this in the form

$$\frac{1}{2}e_1\left\{\frac{e_2}{r_{12}} + \frac{e_3}{r_{13}} + \cdots + \frac{e_n}{r_{1n}}\right\} + \frac{1}{2}e_2\left\{\frac{e_1}{r_{21}} + \frac{e_3}{r_{23}} + \cdots + \frac{e_n}{r_{2n}}\right\} + \cdots$$

The first expression in brackets is just the potential at A due to all the other charges,V_1 say.With a similar meaning for $V_2, \ldots$ the required mutual potential energy is

$$W = 1/2\, e_1V_1 + 1/2\, e_2V_2 + \ldots = \sum 1/2\, e\, V \qquad \ldots(6.24)$$

If the charges are not localized as point charges,but are distributed with volume density ρ and surface density σ the corresponding formula is

$$W = \frac{1}{2}\iiint \rho V\, d\tau + \frac{1}{2}\iint \sigma V dS \qquad \ldots(6.25)$$

The first term may be transformed as follows

$$\frac{1}{2}\iiint \rho V d\tau = \frac{1}{8\pi}\iiint V \operatorname{div} \underline{E}\, d\tau$$

$$= \frac{1}{8\pi}\iiint\left\{\operatorname{div} V\underline{E} - \underline{E}\cdot\operatorname{grad} V\right\}d\tau$$

$$= \iint_S \frac{V E_n}{8\pi}\, dS + \iiint \frac{E^2}{8\pi}\, d\tau$$

The surface integral is taken over the sphere at infinity, where a consideration of orders of magnitude shows that it is zero for any finite set of charges; and over the surface of each conductor where E_n is the normal component of $\underline{E}$ away from τ into each separate conductor. But since $E_n = -4\pi\sigma$, so that

$$\frac{1}{2}\iiint \rho V d\tau + \frac{1}{2}\iint \sigma V ds = \iiint \frac{E^2}{8\pi} d\tau$$

Hence the total electrostatic energy is

$$W = \iiint \frac{E^2}{8\pi} d\tau \qquad \ldots(6.26)$$

6.9 The Electrostatic Stress Tensor:

According to Faraday the tubes of displacement $\underline{D}$ are in a state of stress, it is this stress which gives rise to the energy $(\underline{D}.\underline{E})/8\pi$ and is responsible for the apparent action-at-a-distance beween charges.

For the case of an isotropic homogeneous dielectric,

$$\text{grad}(\underline{E}.\underline{D}) = \underline{E}\wedge(\nabla\wedge\underline{D}) + \underline{D}\wedge(\nabla\wedge\underline{E}) + (\underline{E}.\ \text{grad})\underline{D} + (\underline{D}.\text{grad})\underline{E}$$

But since $\underline{D} = K\,\underline{E}$, where K is a constant, curl $\underline{E}$ = Curl $\underline{D}$ = 0,

Then

$$1/2\ \text{grad}(\underline{E}.\underline{D}) = K(\underline{E}.\text{grad})\underline{E} = (\underline{D}.\text{grad})\underline{E}$$

The x-component of this equation is

$$\frac{1}{2}\frac{\partial}{\partial x}(\underline{E}.\underline{D}) = (\underline{D}.\text{grad})E_1 = \text{div}(E_1\underline{D}) - E_1\,\text{div}\,\underline{D}$$

$$\therefore \frac{\partial}{\partial x}(E_1D_1 - \frac{1}{2}\underline{E}.\underline{D}) + \frac{\partial}{\partial y}(E_1D_2) + \frac{\partial}{\partial z}(E_1D_3) - E_1\,\text{div}\,\underline{D} = 0$$

But if ρ is the density of charge in the medium, div $\underline{D} = 4\pi\rho$, then

$$\rho E_1 = \left\{\frac{\partial}{\partial x}(E_1D_1 - \frac{1}{2}\underline{E}.\underline{D}) + \frac{\partial}{\partial y}(E_1D_2) + \frac{\partial}{\partial z}(E_1D_3)\right\}/4\pi$$

The left hand side of this equation is the x-component of the force exerted by the stress in the medium per unit volume of the charges in the medium. Since the system is in equilibrium, it follows that body forces equal and opposite to this should act on the medium to maintain equilibrium. The x-component of the equation of equilibrium of the medium is thus

$$F_1 + \frac{\partial}{\partial x}\left\{\frac{1}{4\pi}\left(E_1D_1 - \frac{1}{2}\underline{E}\cdot\underline{D}\right) + \frac{\partial}{\partial y}\left(\frac{1}{4\pi}E_1D_2\right) + \frac{\partial}{\partial z}\left(\frac{1}{4\pi}E_1D_3\right)\right\} = 0$$

Therefore the stress matrix is

$$\begin{bmatrix} \frac{1}{4\pi}(E_1D_1 - \frac{1}{2}\underline{E}\cdot\underline{D}) & \frac{1}{4\pi}E_2D_1 & \frac{1}{4\pi}E_3D_1 \\ \frac{1}{4\pi}E_1D_2 & \frac{1}{4\pi}(E_2D_2 - \frac{1}{2}\underline{E}\cdot\underline{D}) & \frac{1}{4\pi}E_3D_2 \\ \frac{1}{4\pi}E_1D_3 & \frac{1}{4\pi}E_2D_3 & \frac{1}{4\pi}(E_3D_3 - \frac{1}{2}\underline{E}\cdot\underline{D}) \end{bmatrix}$$

Inserting $\underline{D}=K\,\underline{E}$, this may be rewritten, after putting $K'=K/4$

$$\begin{bmatrix} K'(E_1^2 - \frac{1}{2}E^2) & K'E_1E_2 & K'E_1E_2 \\ K'E_2E_1 & K'(E_2^2 - \frac{1}{2}E^2) & K'E_2E_3 \\ K'E_3E_1 & K'E_3E_2 & K'(E_3^2 - \frac{1}{2}E^2) \end{bmatrix} \quad \text{.....} (6.27)$$

This is Maxwell's stress system.

If the x-axis is chosen parallel to a line of force at any point, $E_2=E_3=0$ and $E_1=E$, thus

$$\begin{bmatrix} \frac{1}{8\pi}KE^2 & 0 & 0 \\ 0 & -\frac{1}{8\pi}KE^2 & 0 \\ 0 & 0 & -\frac{1}{8\pi}KE^2 \end{bmatrix}$$

which shows that the principal stresses are:

$$T_{11} = \frac{1}{8\pi}KE^2, \quad T_{22} = -\frac{1}{8\pi}KE^2, \quad T_{33} = -\frac{1}{8\pi}KE^2$$

Another way of obtaining the electrostatic stress tensor is as follows. The force per unit volume is $\underline{F}=\rho\underline{E}$, or in tensor notation

$$F_i = \rho E_i$$

But $\quad 4\pi\rho = K \operatorname{div}\underline{E} = \frac{\partial E_j}{\partial x_j}$

so that $\quad 4\pi F_i = KE_i\frac{\partial E_j}{\partial x_j} = K\frac{\partial E_iE_j}{\partial x_j} - K\frac{\partial E_j}{\partial x_j}E_j$

But since $\quad \nabla\wedge\underline{E} = 0, \quad \frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} = 0$

Hence

$$4\pi F_i = K\frac{\partial}{\partial x_j}\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) \quad \text{or} \quad \underline{F_i = \frac{\partial}{\partial x_j}(P_{ij})} \quad \text{.....} (6.28)$$

i.e. the mechanical force may be regarded as derived from stress tensor p_{ij}.

B. MAGNETOSTATICS

6.10 Magnetic Moment and Magnetic Field:

The moment of a magnet is a vector, whose direction is along the axis from the south to north pole and magnitude is proportional to the length l times the pole strength m, i.e.

$$\underline{M} = ml\,\underline{a} \qquad \ldots(6.29)$$

$\underline{a}$ is a unit vector parallel to the axis of the magnet.

Suppose we have a short magnet of length dl placed in a magnetic field of potential Ω, then the potential energy of the magnet is:

$$-m\Omega + m(\Omega + dl\,\underline{a}.\nabla\Omega) = \underline{M}.\nabla\Omega$$

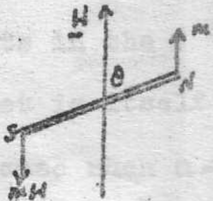
where Ω is the potential at the negative pole and $\Omega + d\Omega$ is that at the positive pole; $d\Omega = dl\,\underline{a}.\nabla\Omega$

The potential energy $W = -\underline{M}.\underline{H}$ $\qquad \ldots(6.30)$

where $\underline{H}$ is the intensity of the field,

$$\underline{H} = -\nabla\Omega \qquad \ldots(6.31)$$

in the neighbourhood of a small magnet.



For a small angular displacement θ, the couple exerted by a small magnet is $G = M\,H\sin\theta$, or

$$\underline{G} = \underline{M} \wedge \underline{H} \qquad \ldots(6.32)$$

The magnetic field of force, at points unoccupied by magnets, is a conservative field and it follows that it is derivable from a potential Ω. Thus, if C is a closed curve, the work done on a unit N-magnetic pole by the magnetic field carried once round C vanishes, so that

$$\oint \underline{H}.d\underline{r} = 0$$

The potential at a field point p may then be defined as" the work done by the magnetic force when a unit N-pole is carried from p to some standard point reckoned as of zero potential", $\underline{H} = -\operatorname{grad}\Omega$.

Since there are no free magnetic poles, the outward flux of the magnetic field across a closed surface, not passing through magnetic matter vanishes, i.e. if $d\underline{S}$ is an element of the closed surface S oriented along the outward drawn normal, then

$$\iint_S \underline{H} \cdot d\underline{S} = \iiint_\tau \operatorname{div} \underline{H}\, d\tau = 0$$

where $d\tau$ is an element of volume. Since this is true for all such surfaces, it follows that at a point unoccupied by magnetic matter,

$$\operatorname{div} \underline{H} = 0 \qquad \ldots(6.33)$$

or $\nabla^2\Omega = 0$. That is, the magnetic potential satisfies Laplace's equation whereas the electric potential satisfies Poisson's equation (6.16).

6.11 Magnetic Field of a Dipole:

If a long bar magnet is cut in half, each half is found to be a magnet. Further subdivision of each half always results in the formation of new magnets so that an element of a magnet is itself a magnet. A magnetic dipole is defined as the limiting case when $m \rightarrow \infty$ and $|\underline{r}| \rightarrow 0$, in such a way that $m|\underline{r}| \rightarrow M$; $|\underline{r}|$ is the magnitude of the vector drawn from the south to the north pole of the small magnet.

The magnetic potential of a dipole of moment $\underline{M}$, situated at a point Q, at a field point p is given by

$$\Omega = \underline{M} \cdot \operatorname{grad}_Q\left(\frac{1}{r}\right)$$

where r is the distance pQ. If $\underline{r} = \underline{Qp}$, then since

$$\operatorname{grad}_Q\left(\frac{1}{r}\right) = \underline{r}/r^3$$

this may be written

$$\Omega = (\underline{M} \cdot \underline{r})/r^3 \qquad \ldots(6.34)$$

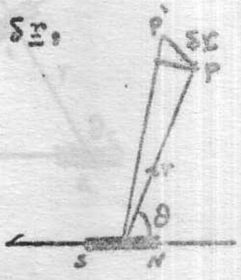
Let θ be the angle which Qp makes with the direction of M, then

$$\Omega = M\cos\theta / r^2 .$$

The corresponding magnetic field $\underline{H} = -\operatorname{grad}\Omega$, is of components:

along the radius vector, $H_r = -\frac{\partial\Omega}{\partial r} = \frac{2M\cos\theta}{r^3}$ ...(6.35)

perpendicular to the radius vector, $H_\theta = -\frac{\partial\Omega}{r\partial\theta} = \frac{M\sin\theta}{r^3}$ ...(6.36)



If a unit N-pole undergoes a small displacement $\underline{pp}' = \delta\underline{r}$, the work done by the magnetic force is

$$\underline{H}\cdot\delta\underline{r} = -\operatorname{grad}\Omega\cdot\delta\underline{r} = -\delta\Omega$$

Thus

$$\underline{H}\cdot\delta\underline{r} = -\delta(\underline{M}\cdot\underline{r})/r^3$$

$$= -\underline{M}\cdot\delta\underline{r}/r^3 + \frac{3(\underline{M}\cdot\underline{r})\,\delta r}{r^4}$$

$$\therefore \quad \underline{H}\cdot\delta\underline{r} = \{-\underline{M}/r^3 + 3(\underline{M}\cdot\underline{r})\underline{r}/r^5\}\cdot\delta\underline{r}$$

Since the displacement pp' is arbitrary, we must have

$$\underline{H} = -\underline{M}/r^3 + 3(\underline{M}\cdot\underline{r})\,\underline{r}/r^5 \qquad ...(6.37)$$

Alternately:

$$\underline{H} = -\operatorname{grad}_p\Omega = -\operatorname{grad}_p\left(\underline{M}\cdot\operatorname{grad}_Q\frac{1}{r}\right)$$

$$= \operatorname{grad}_p\left(\underline{M}\cdot\operatorname{grad}_p\frac{1}{r}\right)$$

$$= -(\underline{M}\cdot\operatorname{grad}_p)(\underline{r}/r^3)$$

$$= -(\underline{M}\cdot\underline{r})\operatorname{grad}_p(1/r^3) - \frac{1}{r^3}(\underline{M}\cdot\operatorname{grad}_p)\underline{r}$$

$$\underline{H} = 3(\underline{M}\cdot\underline{r})\,\underline{r}/r^5 - \underline{M}/r^3$$

since $\underline{M}$ is a constant vector.

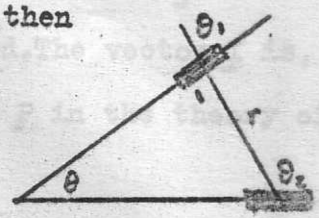
Let $\underline{M}_1$ and $\underline{M}_2$ be the magnetic moments of two dipoles (1&2) and $\underline{H}_2$ be the magnetic field due to the second dipole at the point occupied by the first, we then have

$$\underline{H}_2 = -\underline{M}_2/r^3 + 3(\underline{M}_2\cdot\underline{r})\underline{r}/r^5$$

where $\underline{r}$ is the position vector of the dipole 1 with respect to dipole 2, and thus the mutual potential energy

$$W = -(\underline{M}_1 \cdot H_2) = (\underline{M}_1 \cdot \underline{M}_2)/r^3 - 3(\underline{M}_1 \cdot \underline{r})(\underline{M}_2 \cdot \underline{r})/r^5 \quad \ldots(6.38)$$

Let θ_1 and θ_2 be the angles made by r with the directions of $\underline{M}_1$ and $\underline{M}_2$ respectively. Let θ be the angle made by $\underline{M}_1$ with $\underline{M}_2$, then



$$W = (M_1 M_2 / r^3)(\cos\theta - 3\cos\theta_1 \cos\theta_2)$$

$$= (M_1 M_2 / r^3)(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2)$$

in case of coplanar magnets.

The couple exerted by $\underline{M}_2$ on $\underline{M}_1$ is

$$\underline{G} = \underline{M}_1 \wedge \underline{H}_2$$

$$= -(\underline{M}_1 \wedge \underline{M}_2)/r^3 + 3(\underline{M}_1 \wedge \underline{r})(\underline{M}_2 \cdot \underline{r})/r^5$$

$$= (M_1 M_2 / r^3)(2\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1) \quad \ldots(6.39)$$

The force exerted by $\underline{M}_2$ on $\underline{M}_1$ tends to move the latter without rotation. Suppose $\underline{M}_1$ undergoes a displacement $\delta\underline{r}$ in which the dipole axis remains parallel to itself, then the mutual potential energy W changes to $W + \text{grad } W \cdot \delta\underline{r}$, to the first order. If $\underline{F}_{21}$ is the required force, the increase of energy is $-\underline{F}_{21} \cdot \delta\underline{r}$.

Thus $\underline{F}_{21} \cdot \delta\underline{r} = W - (W + \text{grad } W \cdot \delta\underline{r})$

Since $\delta\underline{r}$ is arbitrary: $\underline{F}_{21} = -\text{grad } W$

Since

$$\text{grad}\left[(\underline{M}_1 \cdot M_2)/r^3\right] = -3(\underline{M}_1 \cdot \underline{M}_2)\,\underline{r}/r^5$$

$$\& \quad \text{grad}\left[(\underline{M}_1 \cdot \underline{r})(\underline{M}_2 \cdot \underline{r})/r^5\right] = -5(\underline{M}_1 \cdot \underline{r})(\underline{M}_2 \cdot \underline{r})\,\underline{r}/r^7 + \left[(\underline{M}_1 \cdot \underline{r})\,\text{grad}(\underline{M}_2 \cdot \underline{r}) + (\underline{M}_2 \cdot \underline{r})\,\text{grad}(\underline{M}_1 \cdot \underline{r})\right]/r^5$$

Then

$$\left.\begin{aligned} \underline{F}_{21} &= (\alpha\,\underline{r} + \beta\,\underline{M}_1 + \gamma\,\underline{M}_2)/r^5 \\ \text{where} \quad \alpha &= 3\left[(\underline{M}_1 \cdot \underline{M}_2) - 5(\underline{M}_1 \cdot \underline{r})(\underline{M}_2 \cdot \underline{r})/r^2\right] \\ \beta &= 3(\underline{M}_2 \cdot \underline{r}) \quad \& \quad \gamma = 3(\underline{M}_1 \cdot \underline{r}) \end{aligned}\right\} \quad \ldots(6.40)$$

6.12 Intensity of Magnetization:

In the molecular theory of magnetism we must suppose that a physically small volume element $\delta\tau$ at a point Q of a magnetized body is a small magnet of dipole $\underline{I}\,\delta\tau$. The vector $\underline{I}$, which is the magnetic moment per unit volume, is termed the intensity of magnetization at Q. If $\underline{I}$ is a constant, the body is said to be uniformly magnetized. The vector $\underline{I}$ is, mathematically, analogous to the polarization vector $\underline{P}$ in the theory of dielectrics.

If P is a point outside a magnetized body, the magnetic potential is given by

$$\Omega = \iiint_\tau \underline{I}\cdot \operatorname{grad}\left(\frac{1}{r}\right)d\tau \qquad \ldots(6.41)$$

In case of a uniformly magnetized body,

$$\Omega = \underline{I}\cdot \iiint \operatorname{grad}_Q (1/r)\, d\tau = -\underline{I}\cdot \iiint_\tau \operatorname{grad}_P (1/r)\, d\tau$$

$$= \underline{I}\cdot(-\operatorname{grad} V)$$

where

$$V \equiv \iiint_\tau d\tau/r$$

is the potential due to a uniform distribution of charge, say, of unit density distributed over the volume. Also if

$$\underline{R} = -\operatorname{grad} V$$

is the intensity due to this distribution at p, we have

$$\Omega = \underline{I}\cdot \underline{R}$$

In the case when the intensity of magnetization is not constant, we have

$$\Omega = \iiint_\tau \underline{I}\cdot \operatorname{grad}_Q (1/r)\, d\tau$$

$$= \iiint_\tau \operatorname{div}_Q (1/r)\, d\tau - \iiint_\tau \frac{1}{r}\operatorname{div}_Q \underline{I}\, d\tau$$

$$= \iint_S \frac{1}{r}\,\underline{I}\cdot d\underline{S} - \iiint_\tau \frac{1}{r}\operatorname{div}_Q \underline{I}\, d\tau \qquad \ldots(6.42)$$

where S is the surface of the magnetized body.

Equation (6.42) expresses the fact that the magnetic field at any point outside the magnetized body is the same as that due to an imaginary volume distribution of magnetic poles of intensity $-\operatorname{div} \underline{I}$ throughout the volume and a surface distribution of magnetic poles of density $\underline{I}.\underline{n}$, where n is a unit vector along the outward normal to the surface of the body S. This is known as Poisson's magnetic distribution.

In the case of a uniformly magnetized body, this distribution reduces to a surface distribution of magnetic poles, since $\operatorname{div} \underline{I} = 0$. Also if θ is the angle which the outward drawn normal to the surface makes with the direction of magnetization, the surface distribution of poles has a density $I \cos \theta$.

6.13 Magnetic Potential & Force inside a Magnetized Body:

Let A be a point inside the magnetized body whose outer surface is S, say. Let C be a cavity surrounding A and let τ_c be its volume, let also Ω_c be the potential at A due to the rest of the magnetized body. As $\tau_c \to 0$, $\Omega_c \to \Omega$ (a unique limit), independent of the shape of the cavity.

The magnetic field $\underline{H}$ at A is

$$\underline{H} = -\operatorname{grad} \Omega$$

Let σ be the surface of the cavity c and I_n the component of the intensity of magnetization $\underline{I}$ along the outward normal to S and σ. Then, if I_Q is the magnetization at a point Q of the body and $r = AQ$,

$$\Omega_c = \iiint_{\tau} \left[\underline{I}_Q \cdot \operatorname{grad}_Q (1/r) \right] d\tau$$

where the volume integral extends over the whole of the magnetized body except the cavity c.

Since $\operatorname{div}(\underline{I}/r) = \frac{1}{r}\operatorname{div}\underline{I} + \underline{I}\cdot\operatorname{grad}(1/r)$

Then $$\Omega_c = \iiint_\tau \left(-\frac{1}{r}\operatorname{div}\underline{I}\right)d\tau + \iint_S \frac{1}{r} I_n\, dS + \iint_\sigma \frac{1}{r} I_n\, d\sigma$$

As $\tau_\sigma \rightarrow 0$, $\Omega_c \rightarrow \Omega$ and the third integral can be ignored, then

$$\Omega = \iiint \left(-\frac{1}{r}\operatorname{div}\underline{I}\right)d\tau + \iint_S \frac{1}{r} I_n\, dS$$

If S_1 is any closed surface drawn entirely within S,

$$\iint_{S_1} \underline{H}\cdot d\underline{S}_1 = -4\pi \iiint_\tau \operatorname{div}\underline{I}\, d\tau$$

where the volume integral extends throughout the region enclosed by S_1.

Using Gauss theorem, we get

$$\iint_{S_1} (\underline{H} + 4\pi\underline{I})\cdot d\underline{S}_1 = 0$$

Introducing the magnetic induction vector B,

$$\underline{B} = \underline{H} + 4\pi\,\underline{I} \qquad \ldots(6.43)$$

Then $$\iint_{S_1} \underline{B}\cdot d\underline{S}_1 = 0 = \iiint_{\tau_1} \operatorname{div}\underline{B}\, d\tau_1$$

Since this volume integral holds for all regions of space enclosed by surface S_1, the integrand vanishes identically. Hence

$$\operatorname{div}\underline{B} = 0 \qquad \ldots(6.44)$$

It thus follows that the lines of magnetic induction are closed curves.

The intensity of magnetization $\underline{I}$ at any point of a magnetized body is directly proportional to the local intensity of the magnetic field $\underline{H}$, i.e. $\underline{I} = k\,\underline{H}$. Since $\underline{B} = \underline{H} + 4\pi\,\underline{I} = (1+4\pi k)\underline{H}$, then we write

$$\underline{B} = \mu\,\underline{H} \qquad \ldots(6.45)$$

$\mu = 1+4\pi k$ is called the permeability of the material.

6.14 Electromagnetic Potentials:

A vector field V is called irrotational(or conservative or lamellar) if curl V=0;in this case V=grad W ,where W(or its negative) is called the Scalar Potential.

If div V=0,the vector field is called solenoidal,in this case V=curl A,where A is a vector function called the vector potential.

In terms of a scalar potential Ø and a vector potential A,the electric field $\underline{E}$ and magnetic field $\underline{B}$ can be expressed by the relations:

$$\underline{E} = -\nabla\varphi - \frac{\partial A}{\partial t} \quad , \underline{B} = \nabla \wedge \underline{A} \qquad \ldots(6.46)$$

On the other hand,Maxwell's equations for free space are:

$$\nabla\wedge\underline{E} = -\frac{1}{c}\frac{\partial H}{\partial t} \quad \& \quad \nabla\cdot\underline{H} = 0 \qquad \ldots(6.47)$$

and

$$\nabla\wedge\underline{H} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + 4\pi\underline{j} \quad \& \quad \nabla\cdot\underline{E} = 4\pi\rho \qquad \ldots(6.48)$$

In order to satisfy (6.47),it is sufficient to put $\underline{H} = \nabla\wedge\underline{A}$. Substitute for $\underline{H}$ and $\underline{E}$ (as in eq.6.46) in (6.48),gives:

$$\nabla\wedge(\nabla\wedge A) = -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\varphi) + 4\pi\underline{j}/c$$

$$= \nabla(\operatorname{div} A) - \nabla^2 A$$

$$\therefore \; -\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \nabla\left(\operatorname{div}\underline{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) = 4\pi\underline{j}/c$$

Let

$$\operatorname{div}\underline{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \qquad \text{Lorentz Condition} \ldots(6.49)$$

then we obtain

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -4\pi\underline{j}/c \qquad \ldots(6.50)$$

But

$$\operatorname{div}\underline{E} = \operatorname{div}\left(-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla\varphi\right) = 4\pi\rho$$

$$\therefore \; -\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\varphi}{\partial t}\right) - \nabla^2\varphi = 4\pi\rho$$

Then

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho \qquad \ldots(6.51)$$

C.HYDRODYNAMICS

6.15 Differentiation following the Motion:

Let us denote any general property of a particle of the fluid, such as its pressure, density, etc., by the function $H(x,y,z,t)$. Then by $\partial H/\partial t$ is meant the variation of H at a particular point in space as a function of the time t. If we take the total differential of $H(x,y,z,t)$, we obtain

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz + \frac{\partial H}{\partial t}dt$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial H}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial H}{\partial z}\cdot\frac{dz}{dt} + \frac{\partial H}{\partial t}$$

The quantity dH/dt is the variation of H when we fix our attention on the same particle of fluid.

The velocity of the particle is given by

$$\underline{V} = \underline{i}\,\frac{dx}{dt} + \underline{j}\,\frac{dy}{dt} + \underline{k}\,\frac{dz}{dt}$$

By definition we have $\nabla H = \underline{i}\,\frac{\partial H}{\partial x} + \underline{j}\,\frac{\partial H}{\partial y} + \underline{k}\,\frac{\partial H}{\partial z}$

Hence

$$\underline{V}\cdot\nabla H = \frac{\partial H}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial H}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial H}{\partial z}\cdot\frac{dz}{dt} \qquad \ldots(6.52)$$

$$\therefore \quad \frac{dH}{dt} = \underline{V}\cdot\nabla H + \frac{\partial H}{\partial t}$$

6.16 The Continuity Equation:

Consider a region of space containing a fluid of density $\rho(x,y,z,t)$. Let τ be the volume inside an arbitrary closed surface S located in this region. Let $Q(t)$ be the mass of fluid inside the volume τ at any instant. Then

$$Q(t) = \iiint_{\tau} \rho\, d\tau \qquad (i)$$

If $\underline{v}$ denotes the velocity of a typical particle of the fluid, the rate at which the mass of fluid inside τ is increasing is

$$\frac{dQ}{dt} = -\iint_{S} \rho\underline{V}\cdot d\underline{s} \qquad (ii)$$

Now, if we differentiate (i) with respect to time and equate the result to (ii), we have $\frac{dQ}{dt} = \iiint_{\tau}\left(\frac{\partial\rho}{\partial t}\right)d\tau = -\iint_{S}(\rho\underline{V})\cdot d\underline{s}$ $\qquad$ (iii)

But, by Gauss's theorem, we have

$$\iint_S (\rho\underline{v})\cdot d\underline{s} = \iiint_\tau \nabla\cdot(\rho\underline{v})\,d\tau$$

Substituting this into (iii), we obtain

$$\iiint_\tau \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v})\right] d\tau = 0$$

Now since the integrand is continuous and the volume τ is arbitrary, we conclude that

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0 \qquad \ldots(6.53)$$

This is the basic equation of hydrodynamics and is known as the equation of continuity.

Since $\nabla\cdot(\rho\underline{v}) = \rho\nabla\cdot\underline{v} + \underline{v}\cdot\nabla\rho$ (Ex.3.12)

Then

$$\frac{d\rho}{dt} + \rho\nabla\cdot\underline{v} = 0 \qquad \ldots(6.54)$$

using the definition of the total derivative (6.52).

If the fluid is incompressible, then ρ is a constant and we have

$$\nabla\cdot\underline{v} = 0$$

If the flow is irrotational, then we have $\nabla\wedge\underline{v}=0$, and we know from sec.3.4 (Ex.3.17) that there exists a scalar function ϕ such that $\underline{v}=\nabla\phi$. Therefore, for incompressible fluid, the velocity potential ϕ satisfies

$$\nabla^2\phi = 0 \qquad \text{Laplace's Equation} \qquad \ldots(6.55)$$

On a fixed boundary of the fluid, the velocity has no normal component, and if $\partial/\partial n$ denotes differentiation with respect to the normal direction of the boundary, we must have $\partial\phi/\partial n = 0$, as a consequence of $\underline{v} = \nabla\phi$.

6.17 Euler's Equation of Motion:

To obtain the equation of motion of a frictionless fluid, we consider the forces acting on an element of fluid whose volume is dxdydz and whose mass is ρ dxdydz. As a consequence of the pressure of the fluid, p, there will be a force in the x direction on the element of fluid under consideration, of magnitude

$$p\,dy\,dz - \left(p + \frac{\partial p}{\partial x}dx\right)dy\,dz = -\frac{\partial p}{\partial x}dx\,dy\,dz$$

Let us also consider the action of an external force $\underline{F}$ acting on unit mass of fluid. The external force acting on the element under consideration in the x diredtion is $F_1\rho$ dxdydz. The acceleration of the element in the x direction is dv_1/dt. Hence, by Newton's second law of motion, we have

$$F_1\rho\,dx\,dy\,dz - \frac{\partial p}{\partial x}dx\,dy\,dz = \frac{dv_1}{dt}\rho\,dx\,dy\,dz$$

By this reasoning, and considering the y and z directions, we obtain Euler's equations of motion:

$$\frac{dv_1}{dt} = F_1 - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad ; \quad \frac{dv_1}{dt} = F_2 - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad ; \quad \frac{dv_3}{dt} = F_3 - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

These three scalar equations may be combined into the single vector eq.

$$\frac{d\underline{v}}{dt} = \underline{F} - \frac{1}{\rho}\nabla p \qquad \ldots(6.56)$$

As a consequence of Eq. (6.52), we have

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\,\underline{v}$$

Now by the vector identity (viii-sec.3.5), we have

$$(\underline{v}\cdot\nabla)\underline{v} = \frac{1}{2}\nabla v^2 + (\nabla\wedge\underline{v})\wedge\underline{v}$$

Hence, by the use of the two equations above, we may write Euler's equation of motion in the following form:

$$\frac{\partial \underline{v}}{\partial t} + \frac{1}{2}\nabla v^2 + (\nabla\wedge\underline{v})\wedge\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p \qquad \ldots(6.57)$$

If the external force is conservative, it has a potential K and we have $\underline{F} = -\nabla K$. Moreover, if the motion of the fluid is irrotational, $\nabla\wedge\underline{v} = 0$ and $\underline{v} = \nabla\phi$. In this case Eq. (6.57) becomes $\nabla\left(\frac{\partial\phi}{\partial t}\right) + \frac{1}{2}\nabla v^2 + \nabla K + \frac{1}{\rho}\nabla p = 0$

or $\nabla w = 0$, where $w = \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + K + \frac{P}{\rho}$ ...(6.58)

Now if $d\underline{r}$ denotes an arbitrary path in the fluid at any instant, we have

$$\nabla w . d\underline{r} = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} dz = dw = 0$$

as a consequence of Eq. (6.58). Hence

$$w = \beta(t)$$

where $\beta(t)$ is an arbitrary function of time, since we have integrated along an arbitrary path in the fluid at any instant. That is, we have

$$w = \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + K + \int \frac{dp}{\rho} = \beta(t) \qquad \text{...(6.59)}$$

Equation (6.59) is known as Bernoulli's equation.

In the special case where ρ is constant and the motion is steady so that $\partial \phi/\partial t = 0$, this equation takes the form

$$\frac{v^2}{2} + K + \frac{P}{\rho} = \beta \qquad \text{...(6.60)}$$

where β is now a constant. This equation states that, "per unit mass of fluid the sum of the kinetic energy, the potential energy K, and the pressure energy P/ρ has a constant value β for all points of the fluid."

Usually in aerodynamics the variations in K are so small that they can be neglected so that we can write

$$\frac{\rho v^2}{2} + P = P_0$$

where p_0 is the pressure when the fluid is at rest. It can be seen from this that, in this case, the pressure diminishes as the velocity increases in the form $P = P_0 - \frac{\rho v^2}{2}$. Airplanes are equipped with instruments for measuring p and p_0 so that the velocity of the machine relative to the air can be determined by the equation $v = \sqrt{2(P_0 - P)/\rho}$

D.DYNAMICS OF A PARTICLE

6.18 Gravitational Potential:

Consider two particles at Q and P of masses m_1 and m_2 respectively. Then, according to the Newtonian law of gravitation, there is a force of attraction between them given in magnitude by the equation

$$F = k\frac{m_1 m_2}{r^2}$$

where k is a constant depending upon the units and r is the distance between the particles.

If we choose the unit of mass as that of a particle which, placed at a unit distance from one of equal mass, aattracts it with unit force, Eq. (i) becomes $F = m_1 m_2/r^2$. If we denote the vector $\underline{QP}$ by $\underline{r}$, we may express the force per unit mass at P due to the attracting particle at Q by

$$\underline{F} = -m\underline{r}/r^3 = \nabla(m/r) \qquad ...(6.61)$$

$\underline{F}$ is called the intensity of force or the gravitational field of force at the point P, and we note that it may be expressed as the gradient of the scalar m/r. It follows therefore that $\underline{F}$ satisfies the equation

$$\nabla\wedge\underline{F} = \nabla\wedge\nabla(m/r) = 0$$

and is therefore a conservative field of force.

Let us suppose that the particle m is stationary at Q and that another of unit mass moves under the attraction of the former from infinity up to P along any path. The work done by the force of attraction during an infinitesimal displacement $d\underline{r}$ of the unit mass is $\underline{F}.d\underline{r}$. The total work done by the force while the unit particle moves from infinity up to P is

$$\int_{\infty}^{P}\underline{F}.d\underline{r} = \int_{\infty}^{P}\nabla(m/r).d\underline{r} = \frac{m}{r}\Big|_{\infty}^{P} = \frac{m}{p}$$

This is independent of the path by which the particle comes to P and is called the potential at P due to the particle of mass at Q.

Let us denote it by V. We then have V=m/P, While the intensity of force at P due to it is

$$\underline{F} = \nabla\left(\frac{m}{P}\right) = \nabla V \qquad ...(6.62)$$

THat is, the intensity at any point is equal to the gradient of the potential.

If we now suppose that there are n particles of masses $m_1, m_2, \dots, m_n$, relative to which p has position vectots $\underline{r}_1, \dots, \underline{r}_n$, respectively, then the force of attraction per unit mass at P due to the system is the vector sum of the intensities due to each, that is,

$$\underline{F} = \nabla\left(\frac{m_1}{r_1}\right) + \nabla\left(\frac{m_2}{r_2}\right) + \text{----} \text{---} + \nabla\left(\frac{m_n}{r_n}\right) = \nabla \sum_{s=1}^{n} \frac{m_s}{r_s}$$

Now, by the same argument as before, the work done by the attracting forces on a particle of unit mass while it moves from infinity up to P is

$$\int_{\infty}^{P} \underline{F} \cdot d\underline{r} = \sum_{s=1}^{n} \frac{m_s}{r_s} = V$$

Therefore the potential at P due to a system of particles is the sum of the potentials due to each. The potential V is a scalar function of position and, except at the points Q, where the masses are situated, it satisfies

$$\nabla^2 V = \nabla^2 \sum_{s=1}^{n} \frac{m_s}{r_s} = \sum_{s=1}^{n} \nabla^2 \frac{m_s}{r_s} = 0 \qquad ...(6.63)$$

which is Laplace's equation. Since $\underline{F} = \nabla V$, this means that at points excluding matter we must have

$$\nabla . \underline{F} = 0 \qquad ...(6.64)$$

For a continuous distribution of matter, the potential at p due to the entire body is given by

$$V_p = \iiint \frac{\rho \, d\tau}{r} \qquad ...(6.65)$$

where r is the distance from the element of volume dV to the point p. If the point p is inside the body, surround p by a closed spherical surface s_o, and consider the potential due to the matter in the space between s_o and s. Then $V_p = \lim_{s_0 \to 0} \iiint_{s_0}^{s} \frac{\rho \, d\tau}{r}$ ...(6.66)

6.19 Gauss's Law of Gravitation:

We have seen that the potential function satisfies $\nabla^2 V=0$, or Laplace's equation, in the region outside matter. Let us now consider the equation satisfied by the potential in a region inside matter. As mentioned in the previous section, we surround the point p by a sphere s_o (of radius a, say). The gravitational field at the point p is the vector sum of the field $\underline{F}_o$ produced by the matter outside s_o and $\underline{F}_i$ produced by the matter inside s_o. That is,

$$\underline{F}_p = \underline{F}_o + \underline{F}_i$$

Thus

$$\nabla\cdot\underline{F}_p = \nabla\cdot\underline{F}_o + \nabla\cdot\underline{F}_i$$

But since $\underline{F}_o$ is produced by matter outside s_o, we have from (6.54),

$$\nabla\cdot\underline{F}_o = 0$$

Consider the sphere of radius a, as $a \to 0$, the intensity at the surface of this sphere is $F_s = \frac{m}{a^2} = \frac{4}{3}\pi\rho a$ in magnitude and in the direction of the inward drawn normal to the surface. By Gauss theorem we have

$$\lim_{a\to 0}\iiint(\nabla\cdot\underline{F}_i)\,d\tau = \lim_{a\to 0}(\nabla\cdot\underline{F})\tfrac{4}{3}\pi a^3$$
$$= \lim_{a\to 0}\iint \underline{F}_s\cdot d\underline{s} = \lim_{a\to 0}\left(-\tfrac{4}{3}\pi\rho a\cdot 4\pi a^2\right)$$

Hence

$$\nabla\cdot\underline{F}_p = \nabla\cdot\underline{F}_i = -4\pi\rho \qquad \ldots(6.67)$$

But $\underline{F}_p = \nabla V_p$, hence the equation satisfied by the potential in a region containing matter is

$$\nabla^2 V = -4\pi\rho = \nabla\cdot\underline{F} \qquad \underline{\text{Poisson's Eq.}} \quad \ldots(6.68)$$

In view of (6.66) we see that we may take

$$V = \iiint\frac{\rho\,d\tau}{r} \qquad \ldots(6.69)$$

as a solution of Poisson's equation.

Apply Gauss theorem to Eq. (6.68), we then have

$$\iint_s \underline{F}\cdot d\underline{s} = \iiint_\tau(\nabla\cdot\underline{F})\,d\tau = -4\pi\iiint\rho\,d\tau \qquad \ldots(6.70)$$

But $\iiint\rho\,d\tau$ is the total mass enclosed by the surface s.

Therefore, the surface integral of the gravitational force F over a closed surface S drawn in the field is equal to -4π times the total mass enclosed by the surface; which is Gauss's Law of Gravitation.

This theorem has many applications in the potential theory and in the theory of electrostatics.

6.20 Application of Tensor Analysis to the Dynamics of a Particle:

Consider a particle of mass M that is moving in space. Let the position of this particle be given by a system of curvilinear coordinates x^r. As the time t varies, the particle will describe a certain curve in space, called the trajectory of the particle. The equation of this curve may be written in the form

$$x^r = x^r(t) \qquad , r = 1,2,3$$

If we transform to another coordinate system $\bar{x}^r$, in accordance with the transformation $\bar{x}^r = \bar{x}^r(x^1, x^2, x^3)$ then the trajectory of the particle may be obtained in terms of t.

Consider the quantity $v^r = dx^r/dt$. In the new coordinates $\bar{x}^r$, the corresponding quantities are

$$\bar{v}^r = \frac{d\bar{x}^r}{dt} = \frac{\partial \bar{x}^r}{\partial x^s} \cdot \frac{dx^s}{dt} = \frac{\partial \bar{x}^r}{\partial x^s} v^s \qquad \ldots(6.71)$$

which are the components of a contravariant vector. In dynamics, this is the velocity vector and its intrinsic derivative, with respect to time, is

$$a^r = \frac{\delta v^r}{\delta t} = \frac{dv^r}{dt} + \left\{ {r \atop mn} \right\} v^m \frac{dx^n}{dt}$$

$$= \frac{d^2 x^r}{dt^2} + \left\{ {r \atop mn} \right\} \frac{dx^m}{dt} \cdot \frac{dx^n}{dt} \qquad \ldots(6.72)$$

where $\left\{ {r \atop mn} \right\}$ are the Christoffel symbols of the coordinate system x^r. If the coordinates are rectangular cartesian, the Christoffel symbols are identical to zero(sec.5.9) and the vector a^r becomes $a^r = \frac{d^2 x^r}{dt^2}$

These are the components of acceleration along the coordinate axes.

If the mass of the particle M remains constant, it is a quantity independent of the coordinate system and the time and is, therefore, *an invariant*. Newton's second law of motion may be expressed in the form

$$F^r = Ma^r$$

where the contravariant vector F^r is the force vector. In curvilinear coordinates this vector is written in the form

$$F^r = M\frac{d^2x^r}{dt^2} + M\begin{Bmatrix} r \\ mn \end{Bmatrix}\frac{dx^m}{dt}\cdot\frac{dx^n}{dt} \qquad \ldots(6.73)$$

If the axes $\bar{x}^r$ are rectangular cartesian, a force whose components in these coordinates are $\bar{F}^r$ and whose point of application is moved through a small displacement $\delta\bar{x}^r$ does an amount of work given by

$$\delta W = \bar{F}^1\delta\bar{x}^1 + \bar{F}^2\delta\bar{x}^2 + \bar{F}^3\delta\bar{x}^3$$

In orthogonal cartesian coordinates the associated vector $\bar{F}_r$ has exactly the same components as $\bar{F}^r$, and the expression for δW becomes

$$\delta W = \bar{F}_r\,\delta\bar{x}^r$$

From the relation connecting $\bar{x}^r$ and x^r it can be seen that δW can also be written in the form

$$\delta W = F_r\,\delta x^r \qquad \ldots(6.74)$$

Since δW is an invariant, it can be concluded that the F_r are the components of a covariant vector. Since the components of F_r in an orthogonal cartesian coordinate system are those of the force vector, it follows that F_r is the covariant force vector in the coordinates x^r. The covariant and contravariant comonents of this vector are related to each other by the following formulas

$$F_r = g_{rs}\,F^s \qquad \& \qquad F^r = g^{rs}\,F_s \qquad \ldots(6.75)$$

If the expression $F_r dx^r$ is a perfect differential, then the force is said to be conservative and a function V may be defined by the equaion

$$V = -\int F_r\,dx^r \qquad \ldots(6.76)$$

The function V is called the force potential. From (6.76) it follows that

$$F_r = -\frac{\partial V}{\partial x^r}$$

The kinetic energy T of a particle is defined as $Mv^2/2$, where v is the magnitude of the velocity. Therefore

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}Mg_{mn}v^m v^n = \frac{1}{2}Mg_{mn}\dot{x}^m\dot{x}^n \qquad \ldots(6.77)$$

where g_{mn} is the metric tensor of the curvilinear coordinates x^m.

Lagrange's Equation:

If T is differentiated partially with respect to $\dot{x}^r$, then

$$\frac{\partial T}{\partial \dot{x}^r} = Mg_{rm}\dot{x}^m$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) = M\left(g_{rm}\ddot{x}^m + \frac{\partial g_{rm}}{\partial x^n}\dot{x}^m\dot{x}^n\right) \qquad (i)$$

Also, if T is differentiated partially with respect to x^r, the result is

$$\frac{\partial T}{\partial x^r} = \frac{1}{2}M\frac{\partial g_{mn}}{\partial x^r}\dot{x}^m\dot{x}^n \qquad (ii)$$

From (i) and (ii) it can be deduced that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = M\left[g_{rm}\ddot{x}^m + \frac{1}{2}\left(\frac{\partial g_{rm}}{\partial x^n} + \frac{\partial g_{rn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r}\right)\dot{x}^m\dot{x}^n\right] = M\left(g_{rm}\ddot{x}^m + [mn,r]\dot{x}^m\dot{x}^n\right)$$

where $[mn,r]$ is the first Christoffel symbol. Therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = Mg_{rs}\left(\ddot{x}^s + \left\{ {s \atop mn} \right\}\dot{x}^m\dot{x}^n\right) \qquad \ldots(6.78)$$

As a consequence of (6.72) it can be seen that the right-hand member of (6.78) is equal to $Mg_{rs}a^s = Ma_r$. Therefore (6.78) may be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = Ma_r = F_r \qquad \text{Lagrange's Eqs.} \ldots(6.79)$$

If the force system is conservative, Lagrange's equations take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial}{\partial x^r}(T-V) = 0 \qquad \ldots(6.80)$$

Problems-VI -ELECTROSTATICS & MAGNETOSTATICS

6.1 : If a spherical conductor, of radius a, with no other conductors in the neighbourhood, is coated with a uniform thickness d of a shell of which K is the specific inductive capacity. Show that the capacity of the conductor is increased in the ratio

$$K(a+d) : ka + d$$

6.2 : O is a fixed point in an uncharged dielectric and the specific inductive capacity K is the same at all points of a line through O but has different values for different lines (K is independent of r, the distance of any point from O). Show that the potential

$$\phi = A + B/r,$$

where A and B are constants, provides a solution of the equation.

6.3 : If half the space between two concentric spheres be filled with solid dielectric, of specific inductive capacity K, the dividing surface between the solid and air being a plane through the centres of the spheres, show that the capacity of the condenser will be the same as if the whole dielectric were of uniform specific inductive capacity $(1+K)/2$.

6.4 : The space between two concentric spherical shells of radii a & b ($a < b$) is filled in one side of a diametral plane with a dielectric K_1 and on the other side with a dielectric K_2. The inner sphere is given a charge Q and the outer is earthed, find the force which acts on the inner sphere and the force which acts on the diametral plane, separating the dielectrics.

6.5 :Two small magnets lie in the same plane and make angles θ_1 and θ_2 with the line joining their centres.Show that the line of action of the resultant force between them divides the lines of centres in the ratio

$$(\tan \theta_2 + 2 \tan \theta_1):(\tan \theta_1 + 2 \tan \theta_2)$$

6.6 :Two magnetic molecules lie in one plane,their axes making angles θ_1 and θ_2 with the line joining the molecules,show that the couple on the second molecule due to the action of the first vanishes if

$$\tan \theta_1 + 2 \tan \theta_2 = 0$$

6.7 :Two dipoles each of moment M are placed at the points(0,0,0) and (0,0,a) are free to move in an external magnetic field of intensity (0,0,-H).Prove that the potential of the dipoles,in the configuration in which the direction cosines of their axes are $(l,m,n),(l',m',n')$, is

$$HM(n+n') + 2 M^2(ll'+mm'-2nn')/a^3$$

6.8 :A magnetic field of intensity H and direction BA,is applied to two small magnets M_1 & $M_2 > 0$ situated at points A & B and resting in equilibrium with their axes along AB.The magnets are deflected,in a plane,through small angles α & β on the same side of AB.Show that the potential energy is approximately

$$\alpha^2(a^{-3}M_1M_2-HM_1/2) + \beta^2(a^{-3}M_1M_2-HM_2/2) + a^{-3}\,\alpha\beta M_1M_2 \qquad ;a=AB$$

Hence prove that the deflected position is stable if

$$a^3H<[(M_1+M_2)-(M_1^2 -M_1M_2+M_2^2)^{1/2}]$$

6.9 :Two small magnets of equal moments are fixed with their axes parallel to the axis of z and in the same direction and with their centres at the points $(\pm a, 0, 0)$. Show that if another small magnet is free to turn about its centre, which is fixed at the point $(0, y, z)$, its axis will rest in the plane $x=0$ and will make with the z axis an angle

$$\tan^{-1} 3yz/(2z^2-a^2-y^2)$$

6.10:Given $\underline{V} = (x^2-yz)\underline{i} - 2yz\,\underline{j} + (z^2-2zx)\underline{k}$, find the vector potential $\underline{A}$ such that $\underline{V} = \nabla\wedge\underline{A}$.

6.11:If the magnetic scalar potential in a region is

$$\Omega = (C/2)\log(x^2+y^2),$$

find the magnetic vector potential, assuming the z-component is everywhere zero.

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